Hopf Bifurcation and Stability Analysis of a Business Cycle Model With Time-Delayed Feedback

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Abstract

In this paper, a business cycle model with time-delayed feedback is investigated. Firstly, we add a time-delayed feedback controller to the business cycle model and propose a new model. Secondly, the linear stability of the model and the local Hopf bifurcation are studied and we derive the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. Besides, the direction of Hopf bifurcation and the stability of bifurcation periodic solutions are studied by adopting the center manifold theorem and the normal form theory. At last, some numerical simulation results are presented to confirm that the controller can effectively increase the stability region of the business cycle model.

Keywords—*Business cycle model,Time-delayed feedback, Stability, Hopf bifurcation, Numerical simulation.*

I. INTRODUCTION

In recent years, with the differential equations have been widely applied to biology, economics and other fields, many scholars have established some models that can reflect the characteristics of the dynamical systems of differential equations [1-3]. Business cycle is also called economic cycle. It refers to the phenomenon that economic expansion and economic contraction occur alternately and repeatedly in economic operation. In the theory of Macroeconomics, the business cycle is characterized by fluctuation in macroeconomic variables, which is caused by the instability of the business systems[4-6]. The dynamics property of a Kaldor-Kalecki business model are studied in [7-9]. In[10], the authors investigated a business model based on Keynesian's theory and first studied Hopf bifurcation for this model with delay. In[11], Jinchen Yu, Mingshu Peng and Caiyan Zhang discuss the stability and bifurcation of the equilibrium by applying the method of multiple scales. However, there few results on the business cycle model with time-delayed feedback control. In this paper, we investigated a business clcle model with time-delayed feedback and analysis the stability and Hopf bifurcation of the model.

In[10], the business cycle model can be described by the following nonlinear differential equations:

$$\ddot{x}(t)+ax(t-\tau)-qx^3(t)=-v\dot{x}^3(t)-v\dot{x}^2(t)-u\dot{x}(t). \tag{1}$$

where $\tau>0$ represents the delay parameter, x is gross national income, the dot is derivative with respect to time t, $0<a\le 1$ denotes the marginal propensity to consume, $0<u\le 1$ and $1/u$ is the Keynesian multiplier, $q>0$ is a fixed interest rate, $v>0$ is capital-output ratio, also known as the accelerator.

In[11], let $\dot{x}(t)=y(t)$, then the Eq. (1) can be changed into the following form:

$$\begin{cases}\dot{x}(t)=y(t),\\ \dot{y}(t)=-ax(t-\tau)-uy(t)+f(x,y).\end{cases} \tag{2}$$

where $f(x,y)=qx^3(t)-vy^2(t)-vy^3(t)$ and the other parameters are definition of this model are the same as model (1).

In this paper, based on the above model (2), we add a time-delayed feedback controller

$k(x(t)-x(t-\tau))$ to it, Hence, we propose a new model as follows:

$$\begin{cases} \dot{x}(t)=y(t), \\ \dot{y}(t)=-ax(t-\tau)-uy(t)+k(x(t)-x(t-\tau))+f(x,y). \end{cases}$$

(3)

where the parameter k is the feedback gain.

The rest of the paper is arranged as follows, the linear stability of the model and the local Hopf bifurcation are studied and the conditions for the stability and the existence of Hopf bifurcation at the equilibrium are derived in Section 2. In Section 3, according to the method of theory and applications of Hopf bifurcation by Hassard, we analysis the direction and stability of bifurcating periodic solutions. In Section 4, the correctness of theoretical analysis are confirmed by some numerical simulation results. At last, some conlusions are obtained in Section 5.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we focus on the problems of the Hopf bifurcation and stability for the system(3). We also derive the sufficient conditions for the the stability and the existence of Hopf bifurcation at the equilibrium point. Obviously, system (3) has the unique equilibrium point $(0,0)$. The linearation of system (3) at $(0,0)$ is

$$\begin{cases} \dot{x}(t)=y(t), \\ \dot{y}(t)=kx(t)-(k+a)x(t-\tau)-uy(t). \end{cases}$$

(4)

Obviously, the correspoding characteristic equation of model (3) at the equilibrium point is as follows

$$\lambda^2+u\lambda+(k+a)e^{-\lambda\tau}-k=0$$

(5)

Lemma 1. When $\tau=0$ is satisfied, the equilibrium point $(0,0)$ of model (3) is locally asymptotically stable.

Proof. When $\tau=0$ is met, Eq. (5) becomes

$$\lambda^2+u\lambda+a=0$$

(6)

Then we have the following conditions:

$$D_1=u>0,\ D_2=a>0$$

(7)

According to the Routh-Hurwitz criteria, all roots of characteristic equation (6) have negative real parts. Hence, when $\tau=0$ hold, the equilibrium point $(0,0)$ of system (3) is locally asymptotically stable.

Lemma 2. Assume that $2ka+a^2>0$, namely $k>-\frac{a}{2}$ is met. Then Eq.(5) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau=\tau_0$, where

$$\omega_0=\sqrt{\frac{-(2k+u^2)+\sqrt{(2k+u^2)^2+4(2ka+a^2)}}{2}}$$

,

$$\tau_0=\frac{1}{\omega_0}\arccos\left(\frac{\omega_0^2+k}{k+a}\right).$$

Proof. Let $\lambda=i\omega\ (\omega>0)$ is a solution of the characteristic equation (5), then

$$-\omega^2+iu\omega+(k+a)(\cos\omega\tau-i\sin\omega\tau)-k=0.$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega^2+(k+a)\cos\omega\tau-k=0 \\ u\omega-(k+a)\sin\omega\tau=0. \end{cases}$$

(8)

From (8) we obtain

$$\omega^4+(2k+u^2)\omega^2-(2ka+a^2)=0.$$

(9)

Hence,

$$\omega=\sqrt{\frac{-(2k+u^2)+\sqrt{(2k+u^2)^2+4(2ka+a^2)}}{2}}$$

$$\tau_j=\frac{1}{\omega_0}\arccos\left(\frac{\omega_0^2+k}{k+a}\right)+\frac{2j\pi}{\omega_0}\quad j=0,1,2,\cdots.$$

Obviously, let $j=0$, then

$$\omega_0=\sqrt{\frac{-(2k+u^2)+\sqrt{(2k+u^2)^2+4(2ka+a^2)}}{2}}$$

(10)

$$\tau_0=\frac{1}{\omega_0}\arccos(\frac{\omega_0^2+k}{k+a})$$

(11)

As a result, when $\tau=\tau_0$, the equation (5) have a pair of purely imaginary roots.

Lemma 3. Let $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ be the root of (6) with $\alpha(\tau_0)=0$ and $\omega(\tau_0)=\omega_0$. When $2ka+a^2>0$ and $2\omega^2+u^2+2k>0$ hold, then the transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1}\Big|_{\tau=\tau_0}>0$ is satisfied.

Proof. By differentiating both sides of Eq. (5) with regard to τ and applying the implicit function theorem, we have

$$\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_0}=\frac{(k+a)\lambda e^{-\lambda\tau_0}}{2\lambda+u-(k+)\tau_0 e^{-\lambda\tau_0}}$$

$$=\frac{(k+a)\omega_0\sin\omega_0\tau_0+i(k+a)\omega_0\cos\omega_0\tau_0}{(u-(k+a)\tau_0\cos\omega_0\tau_0)+i(2\omega_0+(k+a)\tau_0\sin\omega_0\tau_0)}$$

then

$$\mathrm{Re}\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_0}$$

$$=\frac{\omega^2(u^2+2\omega^2+2k)}{(u-(k+a)\tau_0\cos\omega_0\tau_0)^2+(2\omega_0+(k+a)\tau_0\sin\omega_0\tau_0)^2}$$

(12)

Since $2\omega^2+u^2+2k>0$, thus $\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}\Big|_{\tau=\tau_0}>0$. The proof is completed.

Lemma 4. For Eq. (5), when $\tau<\tau_0$, all of his roots have negative real parts. The equilibrium $(0,0)$ is locally asymptotically stable, and system (3) produces a Hopf bifurcation at the equilibrium $(0,0)$ when $\tau=\tau_0$.

By applying the Hopf bifurcation theorem for time-delayed differential equation and the above four lemmas [12] , we have the following consequences.

Theorem 1. For system (3), the following conclusions hold:

a) If $k>-\frac{a}{2}$ and $2\omega^2+u^2+2k>0$ hold, the equilibrium point $(0,0)$ is asymptotically stable for $\tau\in[0,\tau_0)$.

b) If $\tau=\tau_0$, model (3) exhibits a Hopf bifurcation at the equilibrium point $(0,0)$.

c) If $\tau>\tau_0$, then the equilibrium point of system (4) is unstable.

III. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In this section, by using the normal form theory and the center manifold theorem introduced in [13-15], we discuss the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions when $\tau=\tau_0$.

For notational convenience, let $\tau=\tau_0+\mu$, $u(t)=(x_1(t),x_2(t))^T$ and $u_t(\theta)=u(t+\theta)$ for $\theta\in[-\tau,0]$, clearly, $\mu=0$ is Hopf bifurcation value for (3). For initial condition $\varphi(\theta)=(\varphi_1(\theta),\varphi_2(\theta))^T\in C[-\tau,0]$, the system (3) is equivalent to the following Functional Differential Equation (FDE) system

$$\dot{u}(t)=L_\mu u+F(u_t,\mu).$$

(13)

with

$$L_\mu(\varphi)=B_1\varphi(0)+B_2\varphi(-\tau)$$

(14)

and

$$F(\mu,\varphi)=\begin{pmatrix}0\\ q\varphi_1^3(\mathrm{t})-v\varphi_2^2(\mathrm{t})-v\varphi_2^3(\mathrm{t})\end{pmatrix}$$

(15)

where L_μ is the one family of bounded linear operator in $C[-\tau,0]$ and

$$B_1=\begin{pmatrix} o & 1 \\ k & -u \end{pmatrix},\ B_2=\begin{pmatrix} 0 & 0 \\ -(k+a) & 0 \end{pmatrix}.$$

By the Riesz representation theorem $[16]$, there exists a bounded variation function $\eta(\theta,\mu)$ for $\theta\in[-\tau,0]$, such that

$$L_\mu\varphi=\int_{-\tau}^{0}d\eta(\theta,\mu)\varphi(\theta),\varphi\in C.$$

(16)

we can choose

$$\eta(\theta,\mu)=B_1\delta(\theta)+B_2\delta(\theta+\tau).$$

(17)

where $\delta(\theta)$ is a Delta function. For $\varphi\in C([-\tau,0])$, the operators A and R are defined as follow

$$A(\mu)\varphi(\theta)=\begin{cases}\dfrac{d(\varphi(\theta))}{d\theta}, & \theta\in[-\tau,0),\\ \int_{-\tau}^{0}d(\eta(\theta,\mu)\varphi(\theta)), & \theta=0.\end{cases}$$

(18)

$$R(\mu)\varphi(\theta)=\begin{cases}0, & \theta\in[-\tau,0),\\ F(\mu,\varphi), & \theta=0.\end{cases}$$

(19)

Hence, the Eq. (13) can be written as the following form:

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t$$

(20)

Since $\dfrac{du_t}{d\theta}=\dfrac{du_t}{dt}$, then Eq.(20) can be written as

$$\frac{du_t}{dt}=\begin{cases}\dfrac{du_t}{dt}+0, & \theta\in[-\tau,0),\\ L_\mu u_t+F(u_t,\mu), & \theta=0.\end{cases}$$

(21)

For $\psi\in C[0,\tau]$, we define the adjoint operator $A^*(\mu)$ of $A(\mu)$ as

$$A(\mu)*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,\tau],\\ \int_{-\tau}^{0}d(\eta^T(t,0)\psi(-t)), & s=0,\end{cases}$$

(22)

For $\varphi(\theta)\in C[-\tau,0)$ and $\psi\in C[0,\tau]$, define a bilinear inner product

$$<\psi,\varphi>$$
$$=\bar{\psi}^T(0)\varphi(0)-\int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\bar{\phi}(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi.$$

(23)

where $\eta(\theta)=\eta(\theta,0)$.

Let $\mu=0$; To determine the normal form of operator A, we need to calculate the eigenvectors $q(\theta)$ and $q^*(s)$ of A and A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. we can obtain

$$\begin{cases}A(0)q(\theta)=i\omega_0q(\theta)\\ A^*(0)q^*(s)=i\omega_0q^*(s)\end{cases}$$

(24)

Assume that $q(\theta)=Ve^{i\omega_0\theta}$ is eigenvector of $A(0)$ corresponding to $i\omega_0$ and $q^*(s)=DV^*e^{-i\omega_0 s}$ is eigenvector of $A^*(0)$ corresponding to $-i\omega_0$. By direct calculate, we get

$$q(\theta)=Ve^{i\omega_0\theta}=(v_1,v_2)^Te^{i\omega_0\theta}$$
$$=(1,\rho_1)^Te^{i\omega_0\theta}=(1,i\omega_0)^Te^{i\omega_0\theta}$$

(25)

$$q^*(s)=DV^*e^{-i\omega_0 s}=D(v_1^*,v_2^*)^Te^{-i\omega_0 s}$$
$$=D(\rho_2,1)^Te^{-i\omega_0 s}=D(u-i\omega_0,1)^Te^{-i\omega_0 s}$$

(26)

Now, we verify that $<q^*,q>=1$ and

$< q^*, \bar{q} >= 0$. From(23), we obtain

$$\left\langle q^*, q \right\rangle = \overline{q^*}^T q(0) - \int_{\theta=-\tau_0}^{0} \int_{\xi=0}^{\theta} \overline{q^*}^T (\xi-\theta) d\eta(\theta) q(\xi) d\xi.$$

$$= \bar{D}[\overline{V^*}^T V - \int_{\theta=-\tau_0}^{0} \int_{\xi=0}^{\theta} \overline{V^*}^T e^{-i\omega_0(\xi-\theta)} d\eta(\theta) V e^{i\omega_0 \xi} d\xi\}$$

$$= \bar{D}[\overline{V^*}^T V - \int_{\theta=-\tau_0}^{0} \overline{V^*}^T [d\eta(\theta)] \theta e^{-i\omega_0\theta} V]$$

$$= \bar{D}[\overline{V^*}^T V - \tau_0 e^{i\omega_0\theta} \overline{V^*}^T B_2 V]. \quad (27)$$

Let $\bar{D} = [\overline{V^*}^T V - \tau_0 e^{i\omega_0\theta} \overline{V^*}^T B_2 V]^{-1}$, we can get

$< q^*, q >= 1$.

By $\left\langle \psi, A\varphi \right\rangle = \left\langle A\psi, \varphi \right\rangle$, we obtain

$$-i\omega_0 \left\langle q^*, \bar{q} \right\rangle = \left\langle q^*, A\bar{q} \right\rangle = \left\langle A^* q^*, \bar{q} \right\rangle$$
$$= \left\langle -i\omega_0 q^*, \bar{q} \right\rangle = i\omega_0 \left\langle q^*, \bar{q} \right\rangle.$$

(28)

Therfore $< q^*, \bar{q} >= 0$. The proof is completed.

Using the same notations as in Hassard et al. [13] , we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) =< q^*, u_t >,$$
$$W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t) q(\theta)\}.$$

(29)

On the center manifold C_0 ,we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta)$$

(30)

Where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots$$

.

z and $\bar{z}$ are local coordinates for center manifold C_0 in C in the direction of q and $\bar{q}$, respectively. Note that W is real if u_t is real, therefore we only discuss real solutions. Since $\mu = 0$, it is easy to find that

$$\dot{z}(t) = \langle q^*, \dot{\mu}_t \rangle = \langle q^*, (A(0) + R(0))\mu_t \rangle$$
$$= \left\langle q^*, A\mu_t \right\rangle + \left\langle q^*, R\mu_t \right\rangle$$
$$= i\omega_0 z + \overline{q^*}^T f_0(z, \bar{z}).$$

(31)

Let

$$z'(t) = i\omega_0 z + g(z, \bar{z}),$$

(32)

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots,$$

(33)

from (20) and (32), we have

$$\dot{W} = \dot{u}_t - \dot{z} q - \dot{\bar{z}} \bar{q}$$
$$= \begin{cases} AW - 2\,\mathrm{Re}\,\overline{q^*}^T(0) f_0(z,\bar{z}) q(\theta), & \theta \in [-\tau_0, 0], \\ AW - 2\,\mathrm{Re}\{\overline{q^*}^T(0) f_0(z,\bar{z}) q(\theta)\} + f_0(z,\bar{z}), \theta = 0. \end{cases}$$

(34)

Which can be rewritten as

$$\dot{W} = AW + H(z, \bar{z}, \theta)$$

(35)

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots$$

(36)

On the other hand, on C_0 ,

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}$$

(37)

Using (30) and (32) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, we obtain

$$\dot{W} = i\omega_0 W_{20}(\theta) z^2 - i\omega_0 W_{02}(\theta)\bar{z}^2 + \cdots.$$

(38)

Comparing the coefficients of the above equation with those of (35) and (38), we get

$$\begin{cases}(A - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta),\\ AW_{11}(\theta) = -H_{11}(\theta),\\ (A + 2i\omega_0)W_{02}(\theta) = -H_{02}(\theta).\end{cases}$$

(39)

Notice that $u_t = u(t+\theta) = W(z(t), \bar{z}(t), \theta) + zq + \overline{zq}$ and $q(\theta) = (1, \rho_1)^T e^{i\omega_0\theta}$, we get

$$u_t = \begin{pmatrix} x_1(t+\theta) \\ x_2(t+\theta)\end{pmatrix} = \begin{pmatrix} W^{(1)}(z,\bar{z},\theta) \\ W^{(2)}(z,\bar{z},\theta)\end{pmatrix} + z\begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} e^{i\omega_0\theta} + \bar{z}\begin{pmatrix} 1 \\ \bar{\rho}_1 \end{pmatrix} e^{-i\omega_0\theta}.$$

$$\varphi_1(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\varphi_2(0) = z\rho_1 + \overline{z\rho_1} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\varphi_1^2(0) = z^2 + 2z\bar{z} + \bar{z}^2 + [2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)]z^2\bar{z} + \cdots$$

$$\varphi_2^2(0) = \rho_1^2 z^2 + 2\rho_1\bar{\rho}_1 z\bar{z} + \bar{\rho}_1^2 z^2 + [2\rho_1 W_{11}^{(2)}(0) + \bar{\rho}_1 W_{20}^{(2)}(0)]z^2\bar{z}$$

$$\varphi_1^3(0) = 3z^2\bar{z}.$$

$$\varphi_1^3(0) = 3\rho_1^2\bar{\rho}_1 z^2\bar{z}.$$

From the (32) and (33), we obtain

$$f(z,\bar{z}) = \begin{pmatrix} 0 \\ K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z}\end{pmatrix}$$

where $K_1 = -v\rho_1^2$, $K_2 = -2v\rho_1\bar{\rho}_1$, $K_3 = -v\bar{\rho}_1^2$,

$$K_4 = 3q - v[2\rho_1 W_{11}^{(2)}(0) + \rho_1 W_{20}^{(2)}(0) - 3\rho_1^2\bar{\rho}_1].$$

When $\bar{q}^{*T}(0) = \bar{D}(1, \bar{\rho}_2)$, we can get that

$$g(z,\bar{z}) = \bar{q}^{*T}(0) f_0(z,\bar{z}) = \bar{D}(1,\bar{\rho}_2)\begin{pmatrix} 0 \\ K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z}\end{pmatrix} = \bar{D}\bar{\rho}_2(K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z}).$$

In order to get the values of g_{20}, g_{11}, g_{02} and g_{21}. Comparing the cofficients of the above equation with those in (33), we get

$$g_{20} = 2\bar{D}\bar{\rho}_2 K_1, g_{11} = \bar{D}\bar{\rho}_2 K_2,$$
$$g_{02} = 2\bar{D}\bar{\rho}_2 K_3, g_{20} = 2\bar{D}\bar{\rho}_2 K_4.$$

(40)

In order to determine the value of g_{21}, we also need to compute the values of $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$H(z,\bar{z},\theta) = -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z})q(\theta)] = -(g_{20}(\theta)\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(\theta) - (\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(\theta).$$

(41)

Comparing the coefficients with (36), we gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),$$
$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

(42)

When $\theta = 0$, we have

$$H(z,\bar{z},0) = -2\operatorname{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z})q(0)] + f_0(z,\bar{z})$$
$$= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(0)$$
$$-(\bar{g}_{20}(\theta)\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(0)$$
$$+\begin{pmatrix} 0 \\ K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2\bar{z} \end{pmatrix}.$$

Comparing the coefficients with (41), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix},$$
$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} 0 \\ K_2 \end{pmatrix}.$$

(43)

Using (39), (42), we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2.$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T, E_2 = (E_2^{(1)}, E_2^{(2)})^T$.

From the definition of $A(0)$ and (39) , we have

$$\int_{-\tau_0}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) - H_{20}(0),$$
$$\int_{-\tau_0}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0).$$

Notice that

$$(i\omega_0 I - \int_{-\tau_0}^{0} e^{i\omega_0\theta} d\eta(\theta))q(0) = 0$$
$$(-i\omega_0 I - \int_{-\tau_0}^{0} e^{-i\omega_0\theta} d\eta(\theta))\bar{q}(0) = 0.$$

Hence, we can get

$$(2i\omega_0 I - \int_{-\tau_0}^{0} e^{2i\omega_0} d\eta(\theta))E_1 = 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix}$$
$$(\int_{-\tau_0}^{0} d\eta(\theta))E_2 = -\begin{pmatrix} 0 \\ K_2 \end{pmatrix}$$

Therefore, we have

$$\begin{cases} \begin{pmatrix} i2\omega_0 & -1 \\ (k+a)e^{-2i\omega_0\tau_0} & i2\omega_0 + u \end{pmatrix}\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ -a & -u \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = -\begin{pmatrix} 0 \\ K_2 \end{pmatrix} \end{cases}$$

(44)

Then we can get

$$E_1^{(1)} = \frac{2K_1}{(k+a)e^{-2i\omega_0\tau_0} - k - 4\omega_0^2 + 2iu\omega_0}$$
$$E_1^{(2)} = i2\omega_0 E_1^{(1)}.$$

(45)

Similarly, we have

$$E_2^{(1)} = \frac{K_2}{a}$$
$$E_2^{(2)} = 0.$$

(46)

Based on the above analysis, we have the following parameters [17-19]:

$$C_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(0)\}},$$
$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2(\operatorname{Im}\{\lambda'(0)\})}{\omega_0}.$$

(47)

which determine the quantities of bifurcating periodic solution in the manifold at the critical value $\tau = \tau_0$, now we have the following theorem for the system (3) [20-22].

Theorem 2.

a) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical . If $\mu_2 < 0$, the Hopf bifurcation is subcritical .

b) β_2 determines the stability of the bifurcating periodic solution. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$, the bifurcating periodic solutions is unstable.

c) The period of the bifurcating periodic solution is decided by the parameter T_2 . If $T_2 > 0(< 0)$, the period increases(decreases).

IV. NUMERICAL SIMULATION

In this section, some numerical results are presented to confirm the analytical predictions obtained in the previous section. We take the parameters $a = 0.5, u = 0.6, q = 0.01, v = 0.8, k = -0.2.$ By simply computing, we obtained that $\omega_0 = 0.494469,$
$\tau_0 = 2.87564$.

From the above arithmetic in section 2, If we choose $\tau = 2.5 < \tau_0$, the equilibrium point $(0,0)$ of the system (3) is asymptotically stable proved by numerical simulations (see Figs. 1-3.).

If the delay value τ passes through the critical value τ_0, the the equilibrium point $(0,0)$ loses its stability and a Hopf bifurcation occurs, namely, there are periodic solutions bifurcating out from the equilibrium point $(0,0)$ (see Figs.4-6.).

For convenient comparison, we can choose the parameters $a = 0.5, u = 0.6, q = 0.01, v = 0.8, k = 0$, namely for the uncontrolled model (2), we obtained that $\omega_0 = 0.592801, \tau_0 = 1.33507$. When $\tau = 1 < \tau_0$, the equilibrium point $(0,0)$ of the system (2) is asymptotically stable $(0,0)$ (see Figs.7-9.). When $\tau = 1.6 > \tau_0$, and the periodic solutions occur from the equilibrium $(0,0)$ (see Figs. 9-12.).

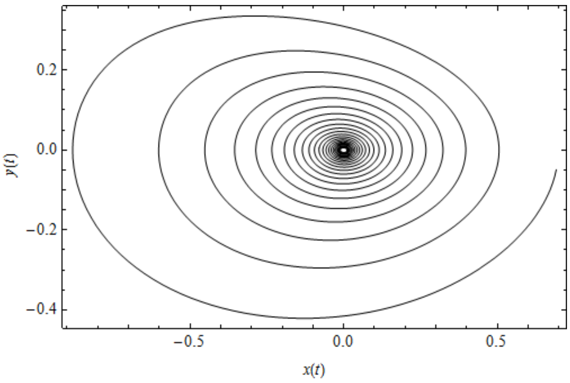

Figure 1. Phase plot of $x(t) - y(t)$ **with** $\tau = 2.5$.

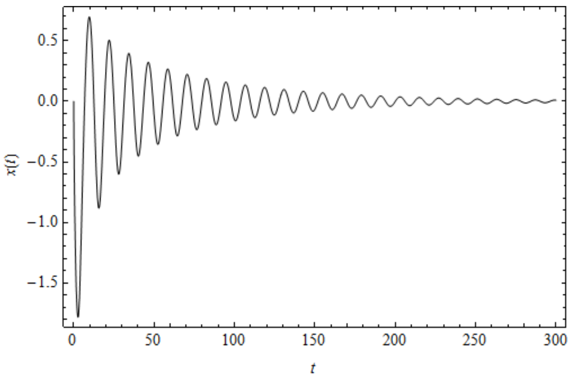

Figure 2. State plot of $x(t)$ **with** $\tau = 2.5$.

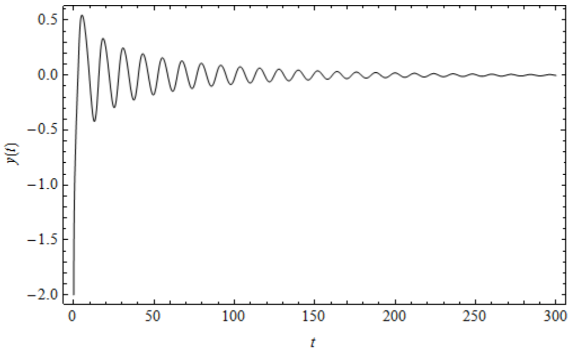

Figure 3. State plot of $y(t)$ **with** $\tau = 2.5$.

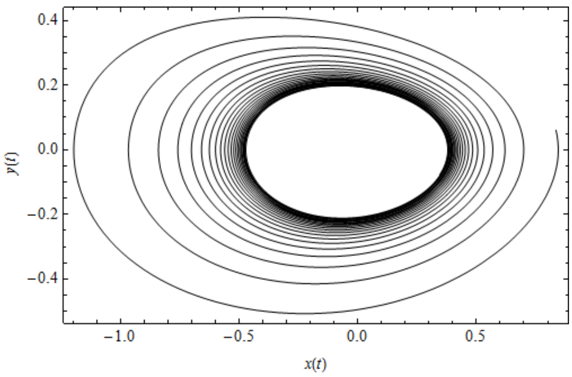

Figure 4. Phase plot of $x(t) - y(t)$ **with** $\tau = 3$.

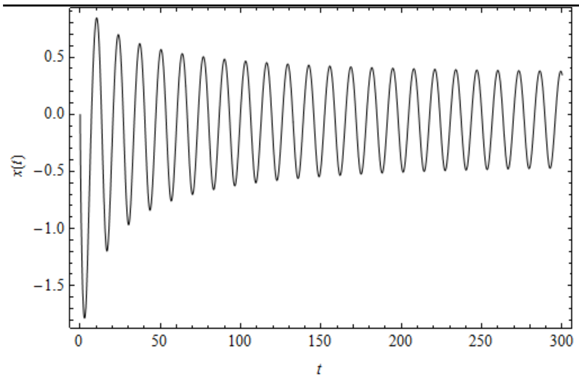

Figure 5. State plot of $x(t)$ **with** $\tau = 3$.

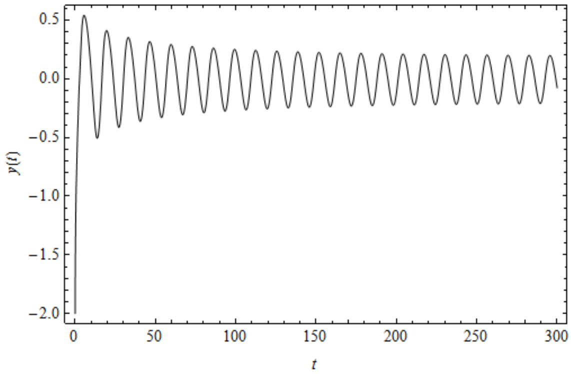

Figure 6. State plot of $x(t)$ **with** $\tau = 3$.

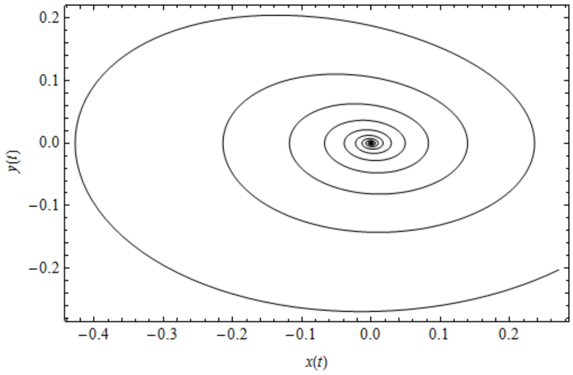

Figure 7. Phase plot of $x(t) - \mathrm{y(t)}$ **with**

$\tau = 1$.

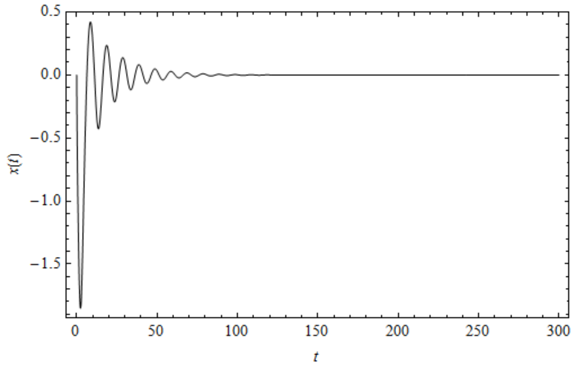

Figure 8. State plot of $x(t)$ **with** $\tau = 1$.

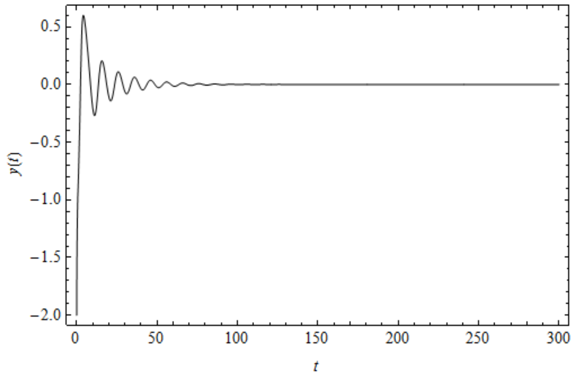

Figure 9. State plot of $y(t)$ **with** $\tau = 1$.

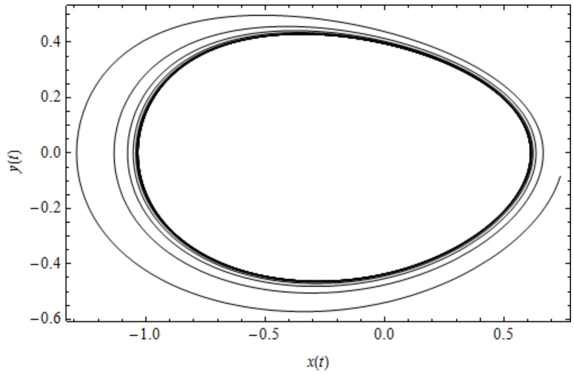

Figure 10. Phase plot of $x(t) - \mathrm{y(t)}$ **with**

$\tau = 1.6$.

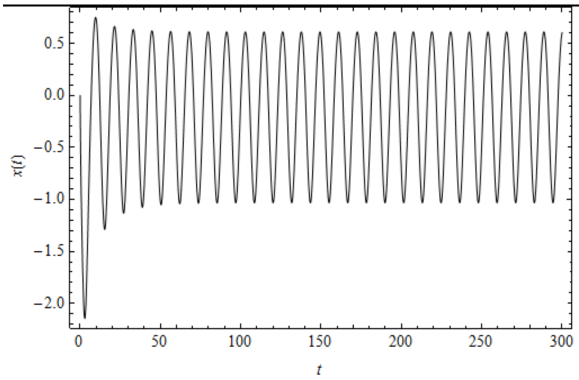

Figure 11. State plot of $x(t)$ **with** $\tau = 1.6$.

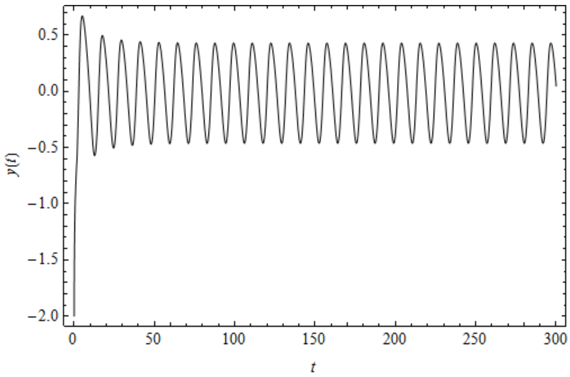

Figure 12. State plot of $y(t)$ **with** $\tau = 1.6$.

V. CONCLUSION

Based on the the control and bifurcation theory, We discussed the effect of the feedbak delay on the system. Until now, there are few results about a business cycle model with feedback delay and we provide an insight to unexplored aspects of them. First, we introduce a time-delayed feedback controller to this model which aim is to control the bifurcation. Second, we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. Moreover, by employing the the center manifold theorem and the normal form theory, we obtained the the direction of Hopf bifurcation and the stability of bifurcation periodic solutions. At last, Some computer simulation results have been presented to illustrate the validity of the theoretical analysis. The research of this paper further enriches and develops the studies on business cycle models.

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