Generalized Hyers-Ulam-Rassias Stability of a Reciprocal Type Functional Equation in Non-Archimedean Fields

M. Sophia[#1]
Associate Professor, Jeppiar Institute of Technology, Department of Mathematics
Tamilnadu, India

Abstract - *In this paper, we obtain the general solution of a reciprocal type functional equation of the type*

$$f(x+y) = \frac{f\left(\frac{3x+2y}{5}\right) f\left(\frac{2x+3y}{5}\right)}{f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right)}$$

And investigate its generalized Hyers-Ulam-Rassias stability in non - Archimedian fields. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias sta-bility and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

I. INTRODUCTION

A significant question concerning the theory of stability of functional equations was raised by S.M. Ulam [29] in 1940 in the University of Wisconsin. In 1941, D.H. Hyers [14] was the first person who presented an a affirmative partial answer to the question of Ulam. In 1950, the theorem formulated by Hyers was generalized by T. Aoki [4] for additive mappings. In 1978, Th.M. Rassias [28] generalized Hyers' theorem which allows the Cauchy difference to be unbounded. In 1982, J.M. Rassias [21] gave a further generalization of the result of D.H. Hyers and proved theorem using weaker conditions controlled by a product of different powers of norms. In 1994, a generalization of Th.M. Rassias' theorem was obtained by P. Gavruta [12] who replaced the unbounded Cauchy difference

by a general control function. In 2008, J.M.

Rassias et.al. [22] discussed the stability of quadratic functional equation

$f(mx + y) + f(mx - y) = 2f(x + y) + 2f(x - y) + 2(m^2 - 2) f(x) - 2f(y)$

for any arbitrary but fixed real constant m with $m\neq 0$; $m\neq \pm 1$; $m\neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [7], [8], [9], [11], [13], [18], [19], [20], [23]). Many research monographs are also available in functional equations, one can see ([1], [2], [3], [10], [15], [16], [17]).

In 2010, K. Ravi and B.V. Senthil Kumar [24] obtained Ulam-Gavruta-Rassias stability for the reciprocal functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1.1}$$

where $f : X \to Y$ is a mapping on the spaces of non-zero real numbers. The reciprocal function $g(x)=\frac{c}{x}$ is a solution of the functional equation (1.1).

K.Ravi, J.M. Rassias and B.V. Senthil Kumar [25] discussed the Ulam stability for the reciprocal functional equation in several variable of the form

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m}\left[\alpha_i \left(\prod_{j=1, j\neq i}^{m} f(x_j)\right)\right]} \tag{1.2}$$

for arbitrary but fixed real numbers $(\alpha_1; \alpha_2; . . . ,\alpha_m) \neq (0; 0; . . . ; 0)$; so that $0 < \alpha = \alpha_1 + \alpha_2 + . . . + \alpha_m = \sum_{i=1}^{m}\alpha_i \neq 1$ and $f : X \to Y$ with X and Y are the spaces of non-zero real numbers.

Later, J.M. Rassias and et.al. [26] introduced the Reciprocal Difference Functional equation(1.3)

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$$

and the Reciprocal Adjoint Functional equation(1.4)

$$f\left(\frac{x+y}{2}\right)+f(x+y)=\frac{3f(x)f(y)}{f(x)+f(y)}$$

and investigated the generalized Hyers-Ulam-Rassias stability of the equations (1.3) and (1.4).

A.Bodaghi and S.O. Kim [5] introduced and studies the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional mapping $f : X \to Y$ satisfying the Rassias quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}. \qquad (1.5)$$

The quadratic reciprocal function $f(x)=\frac{c}{x^2}$ is a solution of the functional equation (1.5). Recently, A. Bodaghi and Y.Ebrahimdoost [6] generalized the equation (1.5) as (1.6)

$$f((a+1)x+ay)+f((a+1)x-ay)=\frac{2f(x)f(y)\,[(a+1)^2f(y)+a^2f(x)]}{((a+1)^2f(y)-a^2f(x))^2}$$

Where $a\in Z$ with $a \neq 0$ and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.6) in non-Archimedean fields.

K.Ravi et al [27] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form (1.7)

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)\left[5r(x)+5r(y)+8\sqrt{r(x)r(y)}\right]}{\left[2r(x)+2r(y)+5\sqrt{r(x)r(y)}\right]^2}$$

In intuitionistic fuzzy normed spaces. In this paper we obtain the general solution of a reciprocal type functional equation of the type (1.8)

$$f(x+y)=\frac{f\left(\frac{3x+2y}{5}\right)f\left(\frac{2x+3y}{5}\right)}{f\left(\frac{3x+2y}{5}\right)+f\left(\frac{2x+3y}{5}\right)}$$

And investigate the generalized Hyers-Ulam-Rassias stability of the equation (1.8) in non-Archimedean fields. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product sum of powers of norms for the equation (1.8).

II. PRELIMINARIES

A non-Archimedean field is a filed A equipped with a function (valuation) | . | from A into $(0,\infty)$ such that for all $r,s\in A$

(i) $|r| = 0$ if and only if $r = 0$

(ii) $|rs| = |r|\,|s|$ and

(iii) $|r+s| \leq \max\{|r|, |s|\}$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n\in N$.

We always assume, in addiition, that | . | is non-trivial, i.e., there exists an

$a_0 \in A$ such that $|a_0| \neq 0, 1$.

An example of a non-Archimedean valuation is the mapping | . |taking every-thing but 0 into 1 and | 0 | = 0. This valuation is called trivial. Another example of a non-Archimedean valuation on a field A is the mapping.

$$|r| \begin{cases} 0 \text{ if } r=0 \\ \frac{1}{r} \text{ if } r>0 \\ 1 \ \text{if} \ldots \end{cases}$$

Let p be a prime number. For any non-zero rational number $x = p^r\frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on Q. It is easy to check that | . | is a non-Archimedean norm on Q. The completion of Q with respect to | . | which is denoted by Q_p is said to be the p-adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n.

III. GENERAL SOLUTION OF EQUATION

Theorem 3.1. Let $f : R^* \to R$ be a function. Then f satisfies (1.1) if and only if f satisfies (1.8). Hence (1.8) is also a reciprocal mapping whose solution is $f(x) = \frac{c}{x}$.

Proof:

Let f satisfy (1.1). Replacing (x,y) by $\left(\frac{3x+2y}{5},\frac{2x+3y}{5}\right)$ in (1.1), we arrive at (1.8). Conversely, suppose f satisfy (1.8). Replacing (x,y) by (3x – 2y, 3y – 2x) in (1.8), we obtain (1.1). This completes the proof of Theorem 3.1.

IV. GENERALIZED HYERS-ULAM STABILITY OF (1.8)

In the following theorems and corollaries, we assume that A and B be a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimedean field A, we put A* - {0}. For convenience, let us define the difference operator D_f: A* X A* $\rightarrow$ B by

$$D_f(x,y) = f(x+y) - \frac{f\left(\frac{3x+2y}{5}\right) f\left(\frac{2x+3y}{5}\right)}{f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right)}$$

For all x,y $\in$ A*. Theorem 4.1. Let ϕ: A* x A* $\rightarrow$ B* be a function such that (4.1)

$$\lim_{n\to\infty} \left|\frac{1}{2}\right|^n \phi\left(\frac{1}{2^{n+1}}x, \frac{1}{2^{n+1}}y\right) = 0$$

For all x,y$\in$ A*. Suppose that f: A* $\rightarrow$ B is a mapping satisfying the inequality

$$|D_f(x,y)| \le \phi(x,y) \tag{4.2}$$

For all x,y$\in$ A*. Then there exists a unique reciprocal mapping r: A* $\rightarrow$ B such that (4.3)

$$|f(x)-r(x)| \le \max\left\{\left|\frac{1}{2}\right|^i \varphi\left(\frac{1}{2^{i+1}}x, \frac{1}{2^{i+1}}y\right) : i \in \mathbb{N}\cup\{0\}\right\}$$

For all x $\in$ A*.

Proof: Replacing (x, y) by (x, x) inb (4.2), we get(4.4)

$$\left|f(2x) - \frac{1}{2}f(x)\right| \le \phi(x,x)$$

For all x$\in$ A*. Now, replacing x by $\frac{x}{2}$ in (4.4) we obtain (4.5)

$$\left|f(x) - \frac{1}{2}f\left(\frac{x}{2}\right)\right| \le \phi\left(\frac{x}{2},\frac{x}{2}\right)$$

For all x$\in$ A*. Plugging x by $\frac{x}{2^n}$ in (4.5) and multiplying by $\left|\frac{1}{2}\right|^n$, we have (4.6)

$$\left|\frac{1}{2^n}f\left(\frac{x}{2^n}\right) - \frac{1}{2^{n+1}}f\left(\frac{x}{2^{n+1}}\right)\right| \le \left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)$$

For all x$\in$ A*. Thus the sequence $\left\{\frac{1}{2^n} f\left(\frac{x}{2^n}\right)\right\}$ is Cauchyby (4.1) and (4.6). Completeness of the non-Archimedean space B allows us to assume that there exists a mapping r so that (4.7)

$$r(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(\frac{x}{2^n}\right)$$

For each x$\in$ A* and non-negative integers n, we have (4.8)

$$\begin{aligned}\left|\frac{1}{2^n}f\left(\frac{x}{2^n}\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1}\left\{\frac{1}{2^{i+1}}f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i}f\left(\frac{x}{2^i}\right)\right\}\right| \\ &\le \max\left\{\left|\frac{1}{2^{i+1}}f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i}f\left(\frac{x}{2^i}\right)\right| : 0 \le i < n\right\} \\ &\le \max\left\{\left|\frac{1}{2}\right|^i \varphi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) : 0 \le i < n\right\}.\end{aligned}$$

Applying (4.7) and letting n $\rightarrow\infty$, we find that the inequality (4.3) holds. From (4.1), (4.2) and (4.7) we have for all x,y $\in$ A*

$$\begin{aligned}|D_r(x,y)| &= \lim_{n\to\infty}\left|\frac{1}{2}\right|^n \left|D_f\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right| \\ &\le \lim_{n\to\infty}\left|\frac{1}{2}\right|^n \phi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0.\end{aligned}$$

Hence the mapping r satisfies (1.8). By Theorem 3.1, the mapping r is reciprocal. Now, let R: A*$\rightarrow$B be another reciprocal mapping satisfying (4.3). Then we have

$$\begin{aligned}|r(x)-R(x)| &= \lim_{p\to\infty}\left|\frac{1}{2}\right|^p \left|r\left(\frac{x}{2^p}x\right) - R\left(\frac{x}{2^p}\right)\right| \\ &\le \lim_{p\to\infty}\left|\frac{1}{2}\right|^p \max\left\{\left|r\left(\frac{x}{2^p}\right) - f\left(\frac{x}{2^p}\right)\right|, \left|f\left(\frac{x}{2^p}\right) - R\left(\frac{x}{2^p}\right)\right|\right\} \\ &\le \lim_{p\to\infty}\lim_{q\to\infty}\max\left\{\max\left\{\left|\frac{1}{2}\right|^{i+p}\phi\left(\frac{x}{2^{i+p+1}},\frac{x}{2^{i+p+1}}\right) : p \le i \le q+p\right\}\right\} \\ &= 0\end{aligned}$$

For all x$\in$ A*, proving that r is unique, which completes the proof.

Theorem 4.2. Let ϕ: A* x A* $\rightarrow$ B* be a function such that (4.9)

$$\lim_{n\to\infty} |2|^n \phi(2^n x, 2^n y) = 0$$

For all x,y$\in$A*. Suppose that f: A* $\to$ B is a mapping satisfying the inequality (4.2) for all x,y$\in$A*. Then there exists a unique reciprocal mapping r: A*$\to$B such that(4.10)

$$|f(x) - r(x)| \leq \max\left\{|2|^{i+1}\phi\left(2^i x, 2^i x\right) : i \in \mathbb{N} \cup \{0\}\right\}$$

For all x$\in$A*.

Proof: Replacing (x,y) by (x,x) in (4.2) and multiplying by |2|, we get (4.11)

$$|2f(2x) - f(x)| \leq |2|\phi(x,x)$$

For all x$\in$A*. Switching x to 2^nx in (4.11) and multiplying by $|2|^n$, we have (4.12)

$$\left|2^n f\left(2^n x\right) - 2^{n+1} f\left(2^{n+1} x\right)\right| \leq |2|^{n+1}\phi\left(2^n x, 2^n x\right)$$

For all x$\in$A*. As n$\to\infty$ in (4.12) and using (4.9), we see that the sequence { 2^n f(2^nx) } is a Cauchy sequence. Since B is complete, this Cauchy sequence converges to a mapping r: A*$\to$B defined by (4.13)

$$r(x) = \lim_{n\to\infty} 2^n f\left(2^n x\right).$$

For each x$\in$A* and non-negative integers n, we have(4.14)

$$\begin{aligned} \left|2^n f\left(2^n x\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1} 2^{i+1} f\left(2^{i+1} x\right) - 2^i f\left(2^i x\right)\right| \\ &\leq \max\left\{\left|2^{i+1} f\left(2^{i+1} x\right) - 2^i f\left(2^i x\right)\right| : 0 \leq i < n\right\} \\ &\leq \max\left\{|2|^{i+1}\phi\left(2^i x, 2^i x\right) : 0 \leq i < n\right\}. \end{aligned}$$

Applying (4.13) and letting n $\to\infty$, we find that the inequality (4.10) holds. From (4.9), (4.2) and (4.13), we have for all x, y$\in$A*.

$$\begin{aligned} |D_r(x,y)| &= \lim_{n\to\infty} |2|^n \left|D_f\left(2^n x, 2^n y\right)\right| \\ &\leq \lim_{n\to\infty} |2|^n \phi\left(2^n x, 2^n y\right) = 0. \end{aligned}$$

Hence the mapping r satisfies (1.8). By Theorem 3.1, the mapping r is reciprocal. Now, let R: A* $\to$B be another reciprocal mapping satisfying (4.10). Then we have

$$\begin{aligned} |R(x) - r(x)| &= \lim_{p\to\infty} |2|^p \left|R\left(2^p x\right) - r\left(2^p x\right)\right| \\ &\leq \lim_{p\to\infty} |2|^p \max\left\{\left|R\left(2^p x\right) - f\left(2^p x\right)\right|, \left|f\left(2^p x\right) - r\left(2^p x\right)\right|\right\} \\ &\leq \lim_{p\to\infty}\lim_{q\to\infty} \max\left\{\max\left\{|2|^{i+p+1}\phi\left(2^{i+p}x, 2^{i+p}x\right) : p \leq i \leq q+p\right\}\right\} \\ &= 0 \end{aligned}$$

For all x$\in$A*, which proves that r is unique.

Corollary 4.3. For any fixed $K_1 \geq 0$ and $\alpha \neq -1$, if f: A* $\to$B satisfies

$$|D_f(x,y)| \leq k_1\left(|x|^\alpha + |y|^\alpha\right)$$

For all x,y$\in$A*, then there exists a unique reciprocal mapping r: A* $\to$B satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{2k_1}{|2|^\alpha}|x|^\alpha, & for\ \alpha < -1 \\ 4k_1|x|^\alpha, & for\ \alpha > -1 \end{cases}$$

For every x$\in$A*.

Proof: the required results are obtained by choosing $\phi(x,y) = k_1\left(|x|^\alpha + |y|^\alpha\right)$, for all x,y$\in$A* in Theorem 4.1 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$ and proceeding by similar arguments as in Theorems 4.1 and 4.2.

Corollary 4.4. Let f: A* $\to$B be a mapping and let there exist real numbers a,b: $\alpha = a + b \neq -1$. Let there exists $k_2 \geq 0$ such that

$$|D_f(x,y)| \leq k_2 |x|^a |y|^b$$

For all x,y$\in$A*. Then there exists a unique reciprocal mapping r: A*$\to$B satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{k_2}{|2|^\alpha}|x|^\alpha, & for\ \alpha < -1 \\ 2k_2|x|^\alpha, & for\ \alpha > -1 \end{cases}$$

For every x$\in$A*.

Proof: Considering ϕ(x,) = k2 $|x|^a$ $|y|^b$, for all x,y$\in$A* in Theorem 4.1 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$, the proof of the corollary is complete.

Corollary 4.5. Let $k_3 \geq 0$ and $\alpha \neq -1$ be real numbers, and f: A* $\to$B be a mapping satisfying the functional inequality

$$|D_f(x,y)| \leq k_3\left(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + \left(|x|^\alpha + |y|^\alpha\right)\right)$$

For all x,y$\in$A*. Then there exists a unique reciprocal mapping r: A* $\to$B satisfying (1.8) and

$$|f(x) - r(x)| \leq \begin{cases} \frac{3k_3}{|2|^\alpha}|x|^\alpha, & for\ \alpha < -1 \\ 6k_3|x|^\alpha, & for\ \alpha > -1 \end{cases}$$

For every x$\in$A*.

Proof: The proof follows immediately by taking $\phi(x,y) = \left(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + \left(|x|^\alpha + |y|^\alpha\right)\right)$ in Theorem 4.1 with $\alpha < -1$ and in Theorem 4.2 with $\alpha > -1$.

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