

Topologies Generated By the Fuzzy Numbers

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Abstract

The α - cuts of fuzzy sets play a dominant role in the analysis of fuzzy sets. Recently the authors studied the properties of Topologies generated by the fuzzy sets. The purpose of this paper is to investigate the Topologies generated by the fuzzy numbers.

Keywords

Triangular fuzzy number, Trapezoidal fuzzy number, Alpha cuts, Topology

1. INTRODUCTION AND PRELIMINARIES

The Extension principle of Zadeh [1] has been mainly applied to computing with so called fuzzy numbers. Topologies generated by the Alpha cuts, strong Alpha cuts, fuzzy subsets and their properties have been already investigated by the authors [2], [3]. In the same way it is interesting to study the Topologies generated by the fuzzy numbers. The following definitions and notations are very useful for the investigations. For the concepts and results that are not given here, one may consult [4].

Definition 1.1

A fuzzy subset A of X is defined by its membership function $\mu_A : X \rightarrow [0, 1]$ which assigns a real number $\mu_A(x)$ in the interval [0, 1] to each element $x \in X$ where the value of μ_A at x shows the membership grade of x in A.

NOTATIONS

A - Fuzzy sets

${}^\alpha A$ - Alpha cut of A

${}^{\alpha+} A$ - Strong Alpha cut of A

T(A) - Collection of all Alpha cuts of A

T⁺(A) - Collection of all strong Alpha cuts of A.

Supp(A) - Support of A

Definition 1.2

The Alpha- cut of A, denoted by ${}^\alpha A$, is the crisp set ${}^\alpha A = \{x \in X : A(x) \geq \alpha\}$ and the strong Alpha-cut of A, denoted by ${}^{\alpha+} A$, is the crisp set ${}^{\alpha+} A = \{x \in X : A(x) > \alpha\}$.

Definition 1.3

A fuzzy number is a fuzzy subset A of the set of all real numbers satisfying the following conditions.

- (i) $A(x) = 1$ for exactly one x.
- (ii) The support $\{x : A(x) > 0\}$ of A is bounded.
- (iii) The Alpha-cuts of A are closed intervals.

Definition 1.4

A Triangular fuzzy number is a fuzzy number A defined by

$$A(x) = \begin{cases} \frac{x-a}{c-a}, & a \leq x \leq c \\ \frac{b-x}{b-c}, & c \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

and can be denoted by $[a, c, b]$ where $a \leq c \leq b$.

Definition 1.5

A Trapezoidal fuzzy number is a fuzzy number A defined by

$$A(x) = \begin{cases} \frac{x-a}{c-a}, & a \leq x \leq c \\ 1, & c \leq x \leq d \\ \frac{b-x}{b-d}, & d \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

and can be denoted by $[a, c, d, b]$ where $a \leq c \leq d \leq b$.

The following lemma shows that the fuzzy members can be represented in a piecewise manner.

Lemma 1.6 [4]

Let A be fuzzy subset of R. Then, A is a fuzzy number if and only if there exists a closed interval [a , b] ≠ φ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ l(x) & \text{for } x \in (-\infty, a) \text{ where } l \text{ is a} \\ r(x) & \text{for } x \in (b, \infty) \end{cases}$$

function from (- ∞ , a) to [0, 1] that is monotonically increasing, continuous from the right, and such that l (x)=0 for x ∈ (- ∞ , w₁) ; r is a function from (b, ∞) to [0,1] that is monotonically decreasing, continuous from the left, and such that r(x)= 0 for x ∈ (w₂, ∞).

2. TOPOLOGIES GENERATED BY THE ALPHA CUTS AND STRONG ALPHA CUTS

Proposition 2.1

If A is a triangular fuzzy number [a ,c, b] then

- (i) T(A) = {R} ∪ { [x, y] : (x-a) (b-c) = (c-a) (b-y), a ≤ x ≤ c and c ≤ y ≤ b }.
- (ii) T⁺(A) = { (a, b), φ } ∪ { (x, y) : (x-a) (b-c) = (c-a) (b-y), a ≤ x ≤ c and c ≤ y ≤ b }.

Proof .

Consider the fuzzy number [a, c, b] .The Alpha-cuts and strong Alpha-cuts are given by ⁰A = R, ¹A={c}, ⁰⁺A = (a, b), ¹⁺A= φ .We know that $A(x) = \frac{x - a}{c - a}$ and $A(y) =$

$$\frac{b - y}{b - c}.$$

Suppose 0 < α < 1, then ^αA={x : A(x) ≥ α}.

Choose x and y such that a ≤ x ≤ c and c ≤ y ≤ b with A(x) = A(y) = α. Then ^αA = [x, c] ∪ [c, y] such that (x - a) (b - c) = (c - a) (b - y) = α.

Thus ^αA = [x, y] and ^{α+}A = (x, y) such that (x - a) (b - c) = (c - a) (b - y).

Then T(A)={R} ∪ { [x, y] : (x - a) (b - c) = (c - a) (b - y), a ≤ x ≤ c and c ≤ y ≤ b }.

T⁺(A) = { (a,b), φ } ∪ { (x, y) : (x - a) (b - c) = (c - a) (b - y), a ≤ x ≤ c and c ≤ y ≤ b }.

The above Proposition can be illustrated by using the following example.

Example 2.2

If A is a triangular fuzzy number [1, 2, 4] with

$$A(x) = \begin{cases} x - 1 , & 1 \leq x \leq 2 \\ \frac{4 - x}{2} , & 2 \leq x \leq 4 \\ 0 , & \text{otherwise} \end{cases}$$

- (i) T(A) = {R} ∪ { [1.5, 3] : (x-a) (b-c) = (c-a) (b-y), 1 ≤ x ≤ 2 and 2 ≤ y ≤ 4 }.
- (ii) T⁺(A) = { (1, 4), φ } ∪ { (1.5, 3) : (x-a) (b-c) = (c-a) (b-y), 1 ≤ x ≤ 2 and 2 ≤ y ≤ 4 }.

Corollary 2.3

If A is a triangular fuzzy number [a, c, b] then T(A) generates the Topology on R and T⁺(A) generates the Topology on (a, b).

Remark 2.4

If A is a triangular fuzzy number [a, c, b] then it is evident that T(A) and T⁺(A) generate different Topologies.

Proposition 2.5

If A is a triangular fuzzy number [a ,c, b] where c is the mid point of [a,b] then

- (i) T(A) = { [c - ε , c + ε] : 0 ≤ 2 ε ≤ b - a } ∪ {R}.
- (ii) T⁺(A) = { (c - ε , c + ε) : 0 ≤ 2 ε ≤ b - a } ∪ { φ , (a, b) }.

Proof

Consider the triangular fuzzy number [a, c, b] where c is the mid point of [a, b]. The Alpha-cuts and strong Alpha-cuts are given by ⁰A= R, ¹A={c}, ⁰⁺A = (a, b), ¹⁺A= φ .

Let 0 < α < 1, by using Proposition 2.1, ^αA = [x, y] ^{α+}A = (x, y) such that (x - a) (b - c) = (c - a) (b - y), a ≤ x ≤ c and c ≤ y ≤ b.

Since c is the mid point of [a, b], x and y are at equal distance from c. So we can choose ε > 0 such that x = c - ε and y = c + ε.

Therefore ^αA = [c - ε , c + ε] and ^{α+}A = (c - ε , c + ε). Since a ≤ x ≤ y ≤ b we have y - x ≤ b - a that implies 2ε ≤ b-a. Therefore

T(A) = { [c - ε , c + ε] : 0 ≤ 2 ε ≤ b - a } ∪ {R} and T⁺(A) = { (c - ε , c + ε) : 0 ≤ 2 ε ≤ b - a } ∪ { φ , (a, b) }.

Proposition 2.6

If A is a trapezoidal fuzzy number [a, c, d, b] then

1. $T(A)=\{R\} \cup \{[x, y], x < y, (x- a) (b - d) = (b - y) (c - a), a \leq x \leq c \text{ and } d \leq y \leq b\}$.
2. $T^+(A)=\{(a, b), \varnothing\} \cup \{(x, y), x < y, (x- a) (b - d) = (b - y) (c - a), a \leq x \leq c \text{ and } d \leq y \leq b\}$.

Proof.

Let $A = [a, c, d, b]$ denotes the trapezoidal fuzzy number. Then ${}^0A=R, {}^1A= [c, d], {}^{0+}A= (a, b), {}^{1+}A= \varnothing$.

Suppose $0 < \alpha \leq 1$, Choose the real numbers x and y such that $a \leq x \leq c$ and $d \leq y \leq b$ with $A(x) = A(y) = \alpha$.

Therefore ${}^aA= [x, y]$ and ${}^{a+}A= (x, y)$ with $(x - a) (b - d) = (b - y) (c - a), a \leq x \leq c$ and $d \leq y \leq b$.

Then $T(A) = \{R\} \cup \{[x, y], x < y, (x - a) (b - d) = (b - y) (c - a), a \leq x \leq c$ and $d \leq y \leq b\}$ and

$T^+(A)=\{(a, b), \varnothing\} \cup \{(x, y), x < y, (x - a) (b - d) = (b - y) (c - a), a \leq x \leq c$ and $d \leq y \leq b\}$.

Proposition 2.7

If A is a trapezoidal fuzzy number [a, c, d, b] with $c - a = b - d$ then

- (i) $T(A) = \{R\} \cup \{[c - \varepsilon, d + \varepsilon], 0 \leq \varepsilon \leq c - a\}$
- (ii) $T^+(A) = \{(a, b), \varnothing\} \cup \{(c - \varepsilon, d + \varepsilon), 0 \leq \varepsilon \leq c - a\}$.

Proof .

Let [a, c, d, b] be a trapezoidal fuzzy number A with $c - a = b - d$. Then Alpha cuts and strong Alpha cuts are given by ${}^0A= R, {}^1A= [c, d], {}^{0+}A= (a, b), {}^{1+}A= \varnothing$. Let $0 < \alpha \leq 1$.

By using Proposition 2.6, ${}^aA = [x, y]$ with $(x - a) (b - d) = (b - y) (c - a), a \leq x \leq c$ and $d \leq y \leq b$.

Since $c - a = b - d$, x and y are at equal distance from [c, d], we can choose $\varepsilon > 0$ such that $x = c - \varepsilon$ and $y = d + \varepsilon$. Therefore ${}^aA = [c - \varepsilon, d + \varepsilon]$,

${}^{a+}A = (c - \varepsilon, d + \varepsilon)$. Since $a \leq x \leq y \leq b, y - x \leq b - a$, that implies $2\varepsilon + d - c \leq b - a$. This proves that $\varepsilon \leq c - a$. Therefore

$T(A) = \{R\} \cup \{[c - \varepsilon, d + \varepsilon], 0 \leq \varepsilon \leq c - a\}$ and $T^+(A) = \{(a, b), \varnothing\} \cup \{(c - \varepsilon, d + \varepsilon), 0 \leq \varepsilon \leq c - a\}$.

Proposition 2.8

If A is the Standard normal curve then

- (i) $T(A) = \{\phi, R, \{0\}\} \cup \{[-x, x] : x = \sqrt{-\log(2\pi\alpha^2)}, 0 < \alpha < \frac{1}{\sqrt{2\pi}}\}$

$$(ii) T^+(A) = \{\phi, R\} \cup \{[-x, x] : x = \sqrt{-\log(2\pi\alpha^2)}, 0 < \alpha < \frac{1}{\sqrt{2\pi}}\}$$

Proof

We know that for the standard normal curve, $A(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$.

Then ${}^0A = {}^{0+}A = R$;

When $\alpha = \frac{1}{\sqrt{2\pi}}, {}^aA = \{0\}, {}^{a+}A = \phi$;

When $\alpha = 1, {}^1A = \{0\}, {}^{1+}A = \phi$;

When $\frac{1}{\sqrt{2\pi}} < \alpha < 1, {}^aA = {}^{a+}A = \phi$;

when $0 < \alpha < \frac{1}{\sqrt{2\pi}},$

${}^aA = {}^{a+}A = [-\beta, \beta]$ where

$$\beta = \sqrt{-\log(2\pi\alpha^2)}.$$

Then $T(A) = \{\phi, R, \{0\}\} \cup \{[-\beta, \beta] : x = \sqrt{-\log(2\pi\alpha^2)}, 0 < \alpha < \frac{1}{\sqrt{2\pi}}\}$

$T^+(A) = \{\phi, R\} \cup \{[-\beta, \beta] :$

$$\beta = \sqrt{-\log(2\pi\alpha^2)}, 0 < \alpha < \frac{1}{\sqrt{2\pi}}\}.$$

Proposition 2.9

Let A be the fuzzy number. Let l and r be the functions as given in the Lemma 1.6. Then

$T(A) = \{X, [a, b]\} \cup \{(x_1, x_2) : l(x_1) = r(x_2), x_1 < a \text{ and } b < x_2\}$.

$T^+(A) = \{(w_1, w_2), \phi\} \cup \{(x_1, x_2) : l(x_1) = r(x_2), x_1 < a \text{ and } b < x_2\}$.

Proof

We know that ${}^aA = \{x : A(x) \geq \alpha\}$. Then Alpha cuts and strong Alpha cuts are given by ${}^0A = R, {}^{0+}A = (w_1, w_2), {}^1A = [a, b], {}^{1+}A = \phi$.

When $0 < \alpha < 1$, then choose x_1, x_2 with $w_1 \leq x_1 < a$ and $b < x_2 \leq w_2$.

Let $l(x_1) = r(x_2) = \alpha$, then

${}^aA = [x_1, a] \cup [a, b] \cup [b, x_2] = [x_1, x_2]$ and ${}^{a+}A = (x_1, x_2)$.

$T(A) = \{X, [a, b]\} \cup \{(x_1, x_2) : l(x_1) = r(x_2), x_1 < a \text{ and } b < x_2\}$.

$T^+(A) = \{(w_1, w_2), \phi\} \cup \{(x_1, x_2) : l(x_1) = r(x_2),$

$x_1 < a$ and $b < x_2$ }.

Remark 2.10

From the above it is clear that $T(A)$ and $T^+(A)$ generate different Topologies. However under some condition they generate the same Topology. It is given in Theorem 2.13.

Lemma 2.11

Let A be a fuzzy subset of X with $\text{Supp}(A) = X$. For each $\alpha \in [0, 1]$, and for each $x \in {}^\alpha A$ there is a $\beta \in [0, 1]$ such that $x \in {}^{\beta+} A \subseteq {}^\alpha A$.

Proof

Suppose $x \in {}^\alpha A$. If $A(x) > \alpha$ then $x \in {}^{\alpha+} A$ and ${}^{\alpha+} A \subseteq {}^\alpha A$.
 If $A(x) = \alpha$, Choose $\epsilon > 0$ such that $A(x) > \alpha - \epsilon = \beta$ that implies $x \in {}^{\beta+} A$.
 Now $\alpha > \alpha - \epsilon = \beta$ that implies $\alpha > \beta$.
 Therefore ${}^{\beta+} A \subseteq {}^\alpha A$ which shows ${}^{\beta+} A \subseteq {}^\alpha A$.
 Therefore $x \in {}^\alpha A$ implies there exist β with $x \in {}^{\beta+} A \subseteq {}^\alpha A$.

Lemma 2.12

Let A be a fuzzy subset of X with $\text{Supp}(A) = X$. For each $\alpha \in [0, 1]$, and for each $x \in {}^{\alpha+} A$ there is a $\beta \in [0, 1]$ such that $x \in {}^\beta A \subseteq {}^{\alpha+} A$.

Proof:

Let $x \in {}^{\alpha+} A$. Then $A(x) > \alpha$. Take $A(x) = \beta$. Choose $\epsilon > 0$ such that $A(x) \geq \alpha + \epsilon = \beta$ that implies $x \in {}^\beta A$. Suppose $y \in {}^\beta A$ then $A(y) \geq \beta > \alpha$ which implies $y \in {}^{\alpha+} A$.
 Hence $x \in {}^\beta A \subseteq {}^{\alpha+} A$.

Theorem 2.13

Let A be a fuzzy subset of X with $\text{Supp}(A) = X$. Then $T(A)$ and $T^+(A)$ generate the same topology on X .

Proof:

Follows from Lemmas 2.11 and 2.12.

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