M^X/G/1Queueing Model with State Dependent Arrival and Server Vacation

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Abstract

A single server queueing model where in customers arrive at the system according to Poisson process with rate λ in batches of random size X has been considered. State dependent mechanism has been shown as an easy approach to combine server vacation model with exhaustive and one – at – a –time discipline. Explicit expressions for the system size generating functions at departure point have been obtained. Special case, for which capacity is finite for M/G/1 model has also been discussed.

Key words: *Server vacation, batch arrivals, single server and state dependent.*

Introduction

The server vacation models have been a long history. During the last few years, the interest in such models has been further enhanced by its applicability in transportation systems, computer system, telecommunication, airline scheduling as well as industrial processes such as production/ inventory systems etc. Several authers have used the decomposition property in M/G/1 model with server vacations :Cooper[11,12],Levy and Yechiali [25], Scholl and Kleinrock [35],Fuhrmann [14], Fuhrmann and Cooper [15,16],Shanthikumar [33,34],Kella and Yechiali [23],Kella [22].Doshi[13], Takagi[36,37], Boxma and Yechialy [5], Madan et.al [29], Madan and Anabosi [30], Madan and abu Al-Rub [31] ,Wang [39], and Wu and Takagi [40] also analysed M/G/1 queueing models with Server vacations.

Server vacation models for the M/G/1 queueing system have been studied by Baba [3,4],Lee and Srinivasan [26] and Jacob and Madhusoodanan [20], Loris Teghem, Jacquelin [28], Choudhury [6,7,9,] Choudhury and Deka[10] ,Ke,J.C. [21].

Recently bulk arrivals queues with vacations have been studied by Li and Tian [27], Monita et al. [32], Arivudainambi and Godhandandaraman [1] and Ayyappan,G.,And Shyamala,S. [2].Harris and Marchal [19] analysed a state-dependent M/G/1 server vacation model. This paper is the extention of the work of Harris and Marchal [19].

Description of the System

We consider the single server queueing system with group arrivals denoted by M^X/G/1. Harris and Marchal [18] analysed the M/G/1 queue in which distribution of the vacations to the server can be considered state –dependent. Their methodology is based on the theory discussed in Gross and Harris [17]for the solution of a departure point,state-dependent service queue in steady state.

We describe the $M^X/G/1$ queueing system with the following assumptions :

(i) customers arrive at the system according to a poisson process with rate λ in groups of random size X , where X has the distribution

$$P_r\{x = n\} = g_n \ (n > 1)$$

And assuming the existence of its generating function

$$C(z) = E[Z^X] = \sum_{n=1}^{\infty} g_n \ z^n \quad (|z| \leq 1)$$

The probability of n customers arrive in an interval of length ` t ‘ is given by

$$K_n(t) = \sum_{m=0}^{n} e^{-\lambda t} \frac{(\lambda t)^m}{m!} g_n^m \quad (n \geq 0)$$

Here g_n^m is the m-fold convolution of $\{g_n\}$ with itself (i.e. the arrivals from a compound poisson process.

We have

$$g_n^{(o)} = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases}$$

(ii) The service is done in a FIFO manner.

(iii) The customers have independent and identically distributed services times with CDF B(t).

Now we proceed on same line as that of Harris and Marchal [18]. The cumulative distribution function (CDF) for the length of a vacation which begins immediately after a service completion when there are n customers in the system, is denoted by V_n (t) in the presence of n customers in the system ,$V_n(o)$ denotes the probabilitity for zero vacation. For j > 0 ,

$V_{ij} = P_r$ { i arrivals occur during a j^{th} cycle }

$$= \int_0^\infty \sum_{m=0}^{i} \frac{e^{-\lambda t} (\lambda t)^m}{m!} \; g_i^m \qquad d[v_{j*} B]\,(t)$$

Where $[v_{j*} B]$ (t) is the convolution of $V_j(t)$ and B(t)

Now the matrix of transition probabilities (t. p.m.) of Markov chain is given by

$$V = \begin{bmatrix} v_{01} & v_{11} & v_{21} & v_{31} & - - - \\ v_{01} & v_{11} & v_{21} & v_{31} & - - - \\ 0 & v_{02} & v_{12} & v_{22} & - - - \\ 0 & 0 & v_{03} & v_{13} & - - - \\ - & - & - & - & \\ - & - & - & - & \\ - & - & - & - & \end{bmatrix}$$

Here V_{i0} is the probability that (i + 1) arrivals occour during a Cycle, given that at least one group arrives during the vacation period, In case of no arrivals during the vacation period the server's vacation will be continued.

Hence,

$$V_{io} = \frac{\alpha}{1 - \int_0^\infty e^{-\lambda t} \quad dv_0\,(t)} \qquad (1)$$

Where,

$$\alpha = \int_0^\infty \sum_{m=0}^{i} \frac{e^{-\lambda\, t} (\lambda\, t)^{(m+1)}}{m+1\,!} \; g_i^{(m+1)} \; d\,[V_0 * B](t)$$
$$- \int_0^\infty e^{-\lambda\, t} \; dV_0\,(t) \int_0^\infty \sum_{m=0}^{i} \frac{e^{-\lambda\, t} \; (\lambda\, t)^{m+1}}{m+1\,!} \; g^{(m+1)} \; dB\,(t)$$

The steady state probability vector, $\pi = \{\pi_n\}$, can be found from transition as

$$\pi_j = \pi_o \; V_{j,1} + \sum_{i=1}^{j+1} \pi_i \; v_{j-i+1,\,i} \qquad n\,;\,(i = 0,1,2 \dots.)$$

We define the generating function

$$\pi\,(z) = \sum_{k=0}^{\infty} \pi^k \; z^k \qquad (2)$$

and $$K_i\,(z) = \sum_{j=0}^{\infty} v_{i,j} \; z^j \qquad (3)$$

Again, ρ_i the expected value of i^{th} row distribution is given by

ρ_i = E (arrivals during a type i cycle)

$= \lambda\bar{g}$ [E(B) + $\bar{v_i}$] (i > 0) ,

Where, $\bar{v_i}$ denotes the mean length of type i vacation and $\bar{g}$ is the mean group size.

If the conditional probability of no arrivals is denoted by T (0) , then

$$\rho_o = \rho - \frac{\bar{v}_o - 1}{T\,(O)}$$

where , $\rho = \lambda\bar{g}$ E(B) and $\bar{v}_o$ is the mean length of type zero vacation.

now the development of the steady-state system size probabilities at departure points is studied by the following theorem.

Theorem- Let { K_i (z) } , (i ≥ 0) be the probability generating functions of the departure point Markov chain of a state dependent M^X /G/1 queue. If K_i (z) can be expressed as a product of two generating functions such that K_i (z)= k(z).A_i (z), and if for some j such that A_i (z)=A(z) (i≥j) , then the generating function is the product of two functions one of which is the generating function of the classical M^X /G/1 queueing model (without vacation).

Proof- from Harris [19], we know that if the stationary distribution for system size exists then π(z) can be written as

$$\pi(z) = \pi_o C(z) + \sum_{i=1}^{\infty} \pi_i z^{i-1} \quad k_i(z)$$

Hence

$$\pi(z) = k(z)\left|\pi_o \frac{c(z)}{k(z)} + \frac{1}{z}\sum_{i=1}^{j-1} \pi_i z^i A_i(z) + \frac{A(z)}{z}\left(\pi(z) - \sum_{i=0}^{j-1} \pi_i z^i\right)\right|$$

Or

$$[z - A(z)K(z)]\pi(z) = K(z)\left|\pi_o z \frac{c(z)}{k(z)} + \sum_{i=1}^{j-1} \pi_i z^i A_i(z) - \left(A(z)\sum_{i=0}^{j-1} \pi_i z^i\right)\right|$$

Or

$$\pi(z) = \frac{\alpha}{z - A(z)\,K(z)}$$

$$= \frac{K(z)\left[z\frac{c(z)}{k(z)} - 1\right]\widehat{\pi_0}}{[z - A(z)\,k(z)]} \quad \frac{1}{\widehat{\pi_0}} \quad \frac{\beta}{[z\frac{C(z)}{K(z)} - 1]}$$

(4)

Where ,

$$\alpha = K(z)\left\{\pi_o \left[z\frac{c(z)}{k(z)} - A(z)\right] + \sum_{i=1}^{j-1} \pi_i z^i [A_i(z) - A(z)]\right\}$$

$$\beta = \pi_o \left[z\frac{c(z)}{K(z)} - A(z)\right] + \sum_{i=1}^{j-1} \pi_i Z^i [A_i(z) - A(z)]$$

And $\widehat{\pi_0}$ is the stationary probability that the M^X /G/1 system is empty with non vacation.

Let $H_i(z)$ be the p.g.f. for the numbers of arrivals in group during a type i (>0) vacation, with $H_i(z)$ = H(z) for all i≥j and $H_o(z)$ denote the conditional generating function for arriving customers in groups during the 'final' vacation of server after idleness.

If i=0 , then from equation (1), we have

$$z[1 - T(o)]\,C(z) = \quad z\,[1 - T(o)] \sum_{i=0}^{\infty} v_{io} z^i$$

$$= \sum_{i=0}^{\infty} \int_0^{\infty} \sum_{m=0}^{i} \frac{e^{-\lambda t} (\lambda t)^{m+1}}{m+1!} z^{i+1} g_i^{(m+1)} [V_0^{*B}](t) - T(0) \sum_{i=0}^{\infty} \int_0^{\infty} \sum_{m=0}^{i} \frac{e^{-\lambda t} (\lambda t)^{m+1}}{m+1!} z^{i+1} g_i^{(m+1)} dB(t)$$

$$= K(z) \quad U_o(z) - T(O)\,K(z)$$

$= k(z)\,[\,U_o(z) - T(O)\,]$

Where) $U_o(z)$ is the unconditional generating function for the number of arrivals during the zero vacation.

Thus we have

$$C(z) = \frac{U_o(z) - T(O)}{z\,[1 - T(o)]} K(z)$$

During the i^{th} cycle , the generating function of the number of arrivals, $K_i(z)$ (i > 0) can be written as

$K_i(z) = K(z)\,H_i(z)$

Or

$$H_0(z) = \frac{U_o(z) - T(O)}{1 - T(o)}$$

And hence

$$C(z) = \frac{H_o(z)}{z} K(z)$$

Now $\pi(z)$ can be expressed as

$$\pi(z) = \frac{K(z)\left[z\frac{C(z)}{K(z)} - 1\right]\widehat{\pi_0}}{[z - H(z)\,K(z)]} \quad \frac{1}{\widehat{\pi_0}} \quad \frac{\theta}{[z\frac{c(z)}{K(z)} - 1]}$$

(5)

Where

$$A_i(z) = \begin{cases} H(z) & ; \quad i \geq j \\ H_i(z) & ; \quad (0 < i < j) \\ \frac{H_0(z)}{z} & ; \quad (i = 0) \end{cases}$$

And

$$\theta = \pi_o [H_o(z) - H(z)] + \sum_{i=1}^{j-1} \pi_i \; z^i [H_i(z) - H(z)]$$

Therefore under the one-at-a-time disciplines, equation (4) becomes

$$\pi(z) = \frac{K(z)[z\,c(z) - k(z)]\,\widehat{\pi_0}}{z - H(z)\,K(z)} \frac{\pi_0}{\widehat{\pi_0}} \left[\frac{\frac{U_0(z) - T(0)}{1 - T(0)} - H(z)}{Z\,C(z) - K(z)} \right] \quad (6)$$

$[H_{i}(z) = H(z) \quad \forall\, i > 0]$

Now since the system is state dependent , second factor of equation (6) leads to

$$\frac{\pi_0}{\widehat{\pi_0}} \frac{1 - U_{0}(z)}{[1 - T(0)]\;[K(z) - z\,c(z)]} \qquad [U_o(z) = H(z)]$$

Now for the exhaustive discipline, we have $H_i(z) = 1$ for $i > 0$,

Hence equation (5) can be written as

$$\pi(z) = \frac{K(z)\;[Z\,C(z) - K(z)]\,\widehat{\pi_0}}{Z - K(z)} \frac{\pi_0}{\widehat{\pi_0}} \quad \frac{H_0(z) - 1}{Z\,C(z) - K(z)} \quad (7)$$

We see that R.H.S. of equation [7] is the product of two terms , first one is the p.g.f. for $M^X/G/1$ [see gross and Harris [17] and second term is due to vacation time.

Conclusion

We have examined the state-dependent server vacation for the $M^X/G/1$ queueing system. The M/G/1 model with vacations to the server considered by Harris and Marchal [18] can also be extended for finite capacity in the similar manner as discussed in this paper.

If N be the finite number of customers for finite M /G/1 system then we get the stationary equation as follow :

$$\pi_i = \begin{cases} \pi_o N_i + \sum_{j=1}^{i+1} \pi_j N_{i-j-1} & ; (i = 0,1,2,\ldots N-2) \\ 1 - \sum_{n=0}^{N-2} \pi_n & ; (i = N-1 \end{cases}$$

We see that the first portion of the equation is same as that of the M/G/1 , therefore

$\pi_i = c\pi_i^*$; (i= 0,1,2,3,…….N-1)

Where π_i and π_i^* are stationary probabilities for $M/G/1/N$ and $M/G/1/\infty$ respectively. And C is given by (see Gross and Harris ,[17]).

$$C = \frac{1}{\sum_{i=0}^{N-1} \pi_i^*}$$

Note that theorem stated in section-2 also holds for finite model. In this case , generating function is defined as ,

$$\pi(z) = \sum_{j=0}^{N-1} c\;\pi_j^*\; z^j$$

$$= c\;\pi_o^* K_o(z) + \sum_{i=1}^{N-1} c\;\pi_i^*\, z^{i-1} K_i(z)$$

The extension of present study to state dependent M^X /G/1 finite model with server vacation is currently the subject of future study.

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