Observer Based Feedback Linearization Control of an Under-actuated Biped Robot

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***Abstract*— This paper deals with the design of a control strategy that combines a nonlinear observer and a partial feedback linearization controller to stabilize the periodic orbits of an under-actuated three link biped robot. We show first that passive dynamics (non-actuated coordinates) can be linearized and decoupled from the rest of the system by applying a nonlinear feedback. Then, the proposed observer is developed in order to estimate the unmeasured velocity signals of the robot. The convergence of the closed loop system is finally proved. Excellent simulations are included to show the effectiveness of the proposed method.**

***Keywords*— Biped robot, control, feedback linearization, observer design.**

I. Introduction

The main complexity in bipedal locomotion is the degree of actuation of the robot. Under-actuated robots are systems that regroup fewer control signals than configuration variables. In literature, several methods are developed to control under-actuated mechanical systems including energy-based approaches [1], linearization technique [2], sliding mode control [3]-[5], flatness based approaches [6], and so on. In [7], Spong shows that it is possible to linearize non-actuated coordinates under a condition called Strong Inertial Coupling. This approach provided good results when applying to acrobat robot. However, most of control strategies suppose knowledge of the all state variables vector. In practice, velocity signals are not always measured due to noise that can affect sensors and consequently it is necessary to use an observer in order to estimate velocities from only position measurements and then construct the control law. In recent years, several types of nonlinear observers have been developed for mechanical systems [8]-[11].

Sliding mode observers have been proposed for many mechanical systems [12]-[14]. In [15], a second-order sliding mode observer is designed to estimate velocities of a switched mechanical system. However, the design of the observer requires the knowledge of uncertainties modelling. A step-by-step second-order sliding mode adaptive observer was applied in [16] for continuous signal reconstruction. By assuming that the absolute angular value is not measured, a nonlinear observer has been studied in [14] for the control of a walking biped. The approach has been successfully applied but the difficult is the finding of the observer gains and the uncertainties modelling.

The purpose of this paper is to design a control strategy for controlling three-link under-actuated bipedal locomotion. This strategy incorporates a novel nonlinear observer to estimate the unmeasured velocities and a feedback linearization based control law. We show first that the passive coordinate of the robot can be linearized and decoupled from the rest of the system by using a nonlinear feedback. After then, the observer is proposed to ensure an asymptotic finite-time estimation of velocity signals which are used in the development of the control law. The stability of the overall closed loop system is proved.

The reminder of this paper is structured as follows. Section 2 displays the model of the under-actuated three links biped robot and underlines physical properties of this kind of mechanical systems. In section 3, we consider the control law based on partial feedback linearization technique. Section 4 is devoted to the observer design and the asymptotic convergence analysis. Section 5 illustrates the main results through simulations applied to the robot under consideration. Finally, some conclusions are included in Section 6.

II. Under-actuated Biped robot model

This section presents the model of under-actuated three-link biped robot. The mechanical system is composed of a torso of mass M_T, hips of mass M_h and two legs that have equal lengths r and masses m with no ankles and no knees (see figure 1). L designs distance between hips and torso. Initially, three degrees of freedom are obtained from this model. The robot is under-actuated since we have two torques applied between the torso and the legs. It was assumed that the walking cycle consists of successive phases of single support (i.e. one leg is touching the ground) with the impact of the swing leg with the ground. Consequently, the mechanical model of the robot under consideration consists of two parts: a swing phase model described by differential equations and an impact model.

A. Swing phase model

The dynamics of the robot during this phase are described by the following mechanical equation

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = B(\theta)u \quad (1)$$

where Vector θ is composed of the angular positions $\theta = [\theta_1 \theta_2 \theta_3]^T$ which are assumed to be only measured. $u \in \Re^2$ represents the applied torques between the two legs. During the swing phase, the matrices M, C, G and B are defined in Appendix 1.

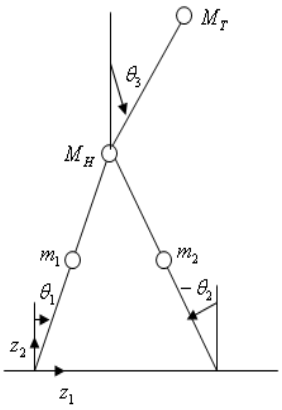

Fig.1 Generalized coordinates of the three link biped robot.

B. Impact model

The contact model is generated when the swing leg touch the walking surface. Thus, two Cartesian coordinates (z_1, z_2) are added to Cartesian coordinates of the end of the stance leg and then we have five degrees of freedom of the biped. The impact is considered as a contact between two rigid bodies. At each impact time, the impact forces are given by impulses. Consequently, jumps are allowed in the velocity signals of the biped robot where configuration variables or angular positions remain constant in each impact. With these considerations, the impact model has the form

$$D_e(q_e).\ddot{q}_e + C_e(q_e,\dot{q}_e).\dot{q}_e + D_e(q) = B_e(q_e).u + \delta F_{ext} \quad (2)$$

where $q_e = [\theta_1, \ \theta_2, \ \theta_3, \ z_1, \ z_2]^T$ is the vector of the generalized coordinates and δF_{ext} is the external forces acting on the biped at the contact point with the surface.

Let the notation $(.)^+$ (resp. $(.)^-$) designs variable or time just after (resp. just before) impact. As mentioned before, during the impact, angular positions remain continuous, i.e. $q_e^{\ +} = q_e^{\ -}$, however, an instantaneous change in the velocities is obtained due to impulsive forces. The relation between the post-impact and the pre-impact can be given by the following relation

$$D_e(q_e).(\dot{q}_e^{\ +} - \dot{q}_e^{\ -}) = F_{ext} \quad (3)$$

where $F_{ext} = \int_{t^-}^{t^+} \delta F_{ext}(\tau) d\tau$. We suppose here that the stance leg detach from the surface of contact without interaction. Thus, the external forces that act the pivot point are equal to zero and for that we consider only the external forces that act at the end of the swing leg. To determine F_{ext}, we should consider constraints on Cartesians coordinates at the end of the swing leg. These constraints are given by:

$$\phi(q_e) = \begin{bmatrix} z_1 + r.\sin(\theta_1) - r.\sin(\theta_2) = d \\ z_2 + r.\cos(\theta_1) - r.\cos(\theta_2) = 0 \end{bmatrix} \quad (4)$$

where d is the length of one biped step. The evaluation of first derivative of (4) gives

$$J(q_e).\dot{q}_e^{\ +} = 0 \quad (5)$$

and

$$F_{ext} = J^T(q).\begin{bmatrix} \lambda_T \\ \lambda_N \end{bmatrix} \quad (6)$$

where J is the Jacobian matrix of constraints given by (4) and λ_T, λ_N are respectively the tangent and normal forces applied at the end of the swing leg. The Jacobian matrix is given by:

$$J(q_e) = \frac{\partial \phi}{\partial q_e} = \begin{bmatrix} r.\cos(\theta_1) & -r.\cos(\theta_2) & 0 & 1 & 0 \\ -r.\sin(\theta_1) & r.\sin(\theta_2) & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

Finally, taking into account these considerations, the impact model corresponds to the following system

$$\begin{cases} M_e(q_e).(\dot{q}_e^{\ +} - \dot{q}_e^{\ -}) = J^T(q_e).\lambda = J^T(q_e).\begin{bmatrix} \lambda_T \\ \lambda_N \end{bmatrix} \\ J^T(q_e).\dot{q}_e^{\ +} = 0 \end{cases} \quad (8)$$

From (8), the post-impact velocity vector is computed as a function of pre-impact velocity vector using the following relation

$$\dot{q}_e^{\ +} = [I - M_e^{\ -1}.J^T.(J.M_e^{\ -1}.J^T)^{-1}J].\dot{q}_e^{\ -} \quad (9)$$

where I is the 3 by 3 identity matrix. All matrices corresponding the impact phase are defined in Appendix 2.

C. State representation

Let $x = (q, \dot{q})$, $x^- = (q^-, \dot{q}^-)$, $x^+ = (q^+, \dot{q}^+)$ and t_i designs an impact time. For each $t \in]t_i, t_{i+1}[$, the continuous dynamics of model (1) can be rewritten as

$$\dot{x} = \begin{bmatrix} \dot{q} \\ M^{-1}(q).[-C(q,\dot{q}).\dot{q} - G(q) + B(q).u] \end{bmatrix} \quad (10)$$
$$:= f(x) + g(x).u$$

This continuous motion has some physical properties (as cited in [10]):

(i) $M(q)$ is a definite-positive-symmetric matrix

(ii) There exists a parameterization for matrix C (.,.) such that $z^T.[\dot{M}(q)/2 - C(q,\dot{q})].z = 0, \forall z \in R^n$,

(iii) The Coriolis and centrifugal forces matrix C(.,.) verify $C(q,x).y = C(q,y).x, \forall (x,y) \in R^n \times R^n$,

For each impact time t_i, we have

$$x^{+}=\begin{pmatrix} q_e^{+} \\ \dot{q}_e^{+} \end{pmatrix}=\Delta(q).x^{-}=\Delta(q)\begin{pmatrix} q_e^{-} \\ \dot{q}_e^{-} \end{pmatrix} \qquad (11)$$

where $\Delta(q)=\begin{pmatrix} R & 0 \\ 0 & R.D(q) \end{pmatrix}$, R is a matrix that corresponds to the legs permutation equal to: $R=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and

$$D(q)=[I-M_e^{-1}.J^T.(J.M_e^{-1}.J^T)^{-1}J].$$

Finally, the overall hybrid model is given by equations (10) and (11), and can be rewritten in the following state-space form

$$\Sigma:\begin{cases} \dot{x}=f(x)+g(x).u\,, & x^{-}\notin S \\ x^{+}=\Delta(x^{-})\quad , & x^{-}\in S \end{cases} \qquad (12)$$

where S is the switching surface (set of points of contact of swing legs with the walking surface).

III. PARTIAL FEEDBACK LINEARISATION BASED CONTROL LAW

Consider the continuous part of the dynamics of the biped robot given by system 10 and assume that the vector of configurations (angular positions) can be partitioned into actuated configurations $q_{ac}=[q_1,q_2]^T$ and non-actuated configurations $q_{nac}=q_3=\theta_3$. Since the robot is underactuated, we assume also that the vector of applied torques is partitioned as $u=[0,u_1]$, where $u_1\in\Re^{2x1}$. Finally, we assume that all matrices defining the biped model are partitioned as follows (see Appendix 3)

$$M(q)=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \text{ where } \begin{cases} M_{11}\in\Re^{1x1}; M_{12}\in\Re^{1x2} \\ M_{21}\in\Re^{2x1}; M_{22}\in\Re^{2x2} \end{cases}$$

$$C(q,\dot{q})=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \text{ where } \begin{cases} C_{11}\in\Re^{1x1}; C_{12}\in\Re^{1x2} \\ C_{21}\in\Re^{2x1}; C_{22}\in\Re^{2x2} \end{cases}$$

$$G(q)=\begin{pmatrix} G_1 & G_2 \end{pmatrix}^T, \text{ where } G_1\in\Re^{1x1}, G_2\in\Re^{2x1}$$

After partitioning all matrices, the model of underactuated biped takes the form of

$$\begin{cases} M_{11}.\ddot{q}_{nac}+M_{12}.\ddot{q}_{ac}+C_{11}.\dot{q}_{nac}+C_{12}.\dot{q}_{ac}+G_1=0 \\ M_{21}.\ddot{q}_{nac}+M_{22}.\ddot{q}_{ac}+C_{21}.\dot{q}_{nac}+C_{22}.\dot{q}_{ac}+G_2=u_1 \end{cases} \qquad (13)$$

Now, we consider the output equation

$$y=q_{ac} \qquad (14)$$

y is called collocated output with the input u_1 [7].

From the definition of the inertia matrix in Appendix 1, since we have $M_{11}=M_T.l^2\neq 0$, we may solve for $\ddot{q}_{nac}$ in equation (13) as

$$\ddot{q}_{nac}=-\frac{1}{M_{11}}[M_{12}.\ddot{q}_{ac}+C_{11}.\dot{q}_{nac}+C_{12}.\dot{q}_{ac}+G_1] \quad (15)$$

Now, when substituting (15) into (13), we get

$$(\overline{M}_{AC}).\ddot{q}_{ac}+(\overline{C}_{NAC}).\dot{q}_{nac}+(\overline{C}_{AC}).\dot{q}_{ac}+(\overline{G})=u_1 \quad (16)$$

with the definition of terms of the following system:

$$\begin{aligned} (\overline{M}_{AC}) &= M_{22}-\frac{M_{21}}{M_{11}}.M_{12} \\ (\overline{C}_{NAC}) &= C_{21}-\frac{M_{21}}{M_{11}}.C_{11} \\ (\overline{C}_{AC}) &= C_{22}-\frac{M_{21}}{M_{11}}.C_{12} \\ (\overline{G}) &= G_2-\frac{M_{21}}{M_{11}}.G_1 \end{aligned} \qquad (17)$$

Since the matrix $(\overline{M}_{AC})$ is positive definite, it is possible to linearize the non-actuated (passive) configuration of the system dynamics. Therefore, the feedback linearization controller is defined for (16) and given by

$$(\overline{M}_{AC}).v_2+(\overline{C}_{NAC}).\dot{q}_{nac}+(\overline{C}_{AC}).\dot{q}_{ac}+(\overline{G})=u_1 \quad (18)$$

where $v_2\in\Re^2$ is an additional control input to be defined later. The closed loop system is then given by:

$$\begin{cases} \ddot{q}_{nac}=-\frac{1}{M_{11}}[M_{12}.v_2+C_{11}.\dot{q}_{nac}+C_{12}.\dot{q}_{ac}+G_1] \\ \ddot{q}_{ac}=v_2 \\ y=q_{ac} \end{cases} \qquad (19)$$

It is clearly seen from (19) that the vector of actuated configurations is completely decoupled from the vector of non-actuated configurations and linearized second order. Let $y_d=[y_{1d}\quad y_{2d}]^T=[q_{1d}\quad q_{2d}]^T$ be the vector of reference trajectories $v_2{*}=\ddot{y}_d=[v_{2,1}{*},v_{2,2}{*}]^T$, and $e=[e_1\quad e_2]^T$, the tracking error vector where each component is given by;

$$e_i=q_i-q_{id}\quad ,i=1,2 \qquad (20)$$

Now, the additional control input $v_2\in\Re^2$ may be chosen component by component as

$$v_{2,i}=v_{2,i}{*}-(k_{i,0}.e_i+k_{i,1}.\dot{e}_i)\quad ,i=1,2 \qquad (21)$$

where coefficients $k_{i,j}, i=1,2, j=0,1$, are chosen so that the two polynomials $\ddot{s}-k_{i,0}s-k_{i,1}\dot{s}, i=1,2$ are Hurwitz. Then, the error system dynamics can be given by:

$$\ddot{e}=v_2-v_2{*} \qquad (22)$$

where each component of (22) is given by :

$$\ddot{e}_i = -(k_{i,0}.e_i + k_{i,1}.\dot{e}_i) \quad ,i=1,2 \tag{23}$$

From (23), we can clearly seen that for a suitable choice of coefficients $k_{i,j}, i=1,2, j=0,1$, the error tracking vector converges globally exponentially to zero.

Let $Z_1 = e$, $Z_2 = \dot{e}$, $\eta_1 = q_{nac} = q_3, \eta_2 = \dot{q}_{nac} = \dot{q}_3$. So, the complete closed loop system can be written as

$$\begin{cases} \dot{Z}_1 = Z_2 \\ \dot{Z}_2 = -K_0.Z_1 - K_1.Z_2 \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \Omega(Z,\eta,t) \\ e = y - y_d = Z_1 \end{cases} \tag{24}$$

Where

$$\Omega(Z,\eta,t) = -\frac{1}{M_{11}}C_{11}.\eta_2 - \frac{1}{M_{11}}[C_{12}.\dot{q}_{ac} + G_1] - \frac{1}{M_{11}}M_{12}.[\ddot{y}_d - K_0.Z_1 - K_1.Z_2] \tag{25}$$

where $K_0 = \begin{pmatrix} k_{10} & 0 \\ 0 & k_{20} \end{pmatrix}$, and $K_1 = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{21} \end{pmatrix}$.

In matrix form, system (24) can be rewritten as

$$\begin{cases} \dot{Z} = A.Z \\ \dot{\eta} = w(Z,\eta,t) \\ e = C.Z \end{cases} \tag{26}$$

where

$Z^T = [Z_1^T \quad Z_2^T]$, $\eta^T = [\eta_1^T \quad \eta_2^T]$, $A = \begin{pmatrix} 0 & I_{2x2} \\ -K_0 & -K_1 \end{pmatrix}$,

$C = [I_{2x2}, 0]$, and $w(Z,\eta,t) = \begin{pmatrix} \eta_2 \\ \Omega(Z,\eta,t) \end{pmatrix}$.

The zero dynamics relative to the output $e = y - y_d$ is given by

$$\dot{\eta} = w(0,\eta,t) \tag{27}$$

Or, when replacing Z by zero into (25),

$$\dot{\eta} = \begin{pmatrix} \eta_2 \\ -\frac{1}{M_{11}}C_{11}.\eta_2 - \frac{1}{M_{11}}[C_{12}.\dot{q}_{ac} + G_1] - \frac{M_{12}}{M_{11}}.\ddot{y}_d \end{pmatrix} \tag{28}$$

which is locally stable for the equilibriums (0,0) and $(0, \pi)$.

IV. OBSERVER DESIGN

Let $\hat{x}_1(t), \hat{x}_2(t) \in \Re^{2x3}$ denote the estimated position and velocity of system (1), and the estimation errors $e_1(t), \dot{e}_1(t), e_2(t) \in \Re^3$ be defined, respectively, by

$$e_1(t) = \hat{x}_1(t) - x_1(t) \tag{29}$$

$$e_2(t) = \hat{x}_2(t) - x_2(t) \tag{30}$$

Let the signal $r(t) \in \Re^n$ be the sliding surface and defined as

$$r(t) = \alpha.e_1(t) + e_2(t) \tag{31}$$

where α is a positive scalar to be chosen, under assumption 1, so that $sgn(r(t)) = sgn(e_1(t)), \forall t \geq 0$, where sgn(.) is the standard signum function.

Assumption1. The initial conditions of the state vector of the mechanical system (12) $[q^T(t_0) \quad \dot{q}^T(t_0)]^T$ and the control force u(t) are chosen so that the position and the velocity vector are bounded functions of time.

By this assumption, and in order to guarantee that $sgn(r(t)) = sgn(e_1(t)), \forall t \geq 0$, the positive scalar α can be chosen such that $\alpha > \frac{\rho_2}{\rho_1}$ where ρ_1, ρ_2 are two positive constants ρ_1, ρ_2 given by $\rho_1 > \|e_1\|_{-\infty} = \min_{1\leq i\leq n}|e_{1,i}|$ and $\rho_2 > \|e_2\|_{+\infty} = \max_{1\leq i\leq n}|e_{2,i}|$. Indeed, using the equation (31), we have $r(t) = \alpha.e_1(t) + e_2(t)$. Then to make $sgn(r(t)) = sgn(e_1(t)), \ \forall t \geq 0$, we have to proceed as following:

- if $e_1 > 0 \Rightarrow r(t)$ must be > 0 which means that $r(t) = \alpha e_1(t) + e_2(t) > 0 \Rightarrow \alpha > -e_2(t)/e_1(t)$

and

- if $e_1 < 0 \Rightarrow r(t)$ must be < 0 which means that $r(t) = \alpha e_1(t) + e_2(t) < 0 \Rightarrow \alpha > -e_2(t)/e_1(t)$ (because we divide by a negative term $e_1 < 0$).

Which, finally, gives $\alpha > \left\| e_2(t)/e_1(t) \right\|_{max}$. Then, after an appropriate choice of the scalar α, we can guarantee that $sgn(r(t)) = sgn(e_1(t)), \forall t \geq 0$. Moreover, if S designs the set of points (e_1, e_2) such that the sliding surface r = 0, then $S = \{0,0\}$.

Now, it is assumed that the velocity signal $v(t) = x_2(t) = \dot{q}(t)$ of the hybrid system (12) is not measured and the only available signal is the output vector $y(t) = x_1(t) = q(t)$. Let the signal $w(t) \in \Re^3$ be equal to $\dot{\hat{x}}_1(t)$. Our objective is to reconstruct both the continuous and discrete states of the hybrid system (12), i.e. to ensure an asymptotic convergence of $(e_1(t), e_2(t))$ to $(0,0)$ as $t \to \infty$. To this end, we propose the following nonlinear observer having y(t) as input vector, w(t) as state vector and the estimated velocity signal $\hat{v}(t)$ as output vector:

$$\begin{cases} \dot{w}(t) = -M^{-1}(y).[C(y,\hat{v}).\hat{v} + G(y) - B(y).u] \\ \qquad -\beta_2.e_1 - \beta_3 \operatorname{sgn}(e_1) \\ \hat{v}(t) = w(t) - (\alpha+\beta_1).e_1(t) \\ w(t_i^+) = Z(e, y(t_i)).[w(t_i^-) - (\alpha+\beta_1).e_1(t_i^-)] \\ \qquad +(\alpha+\beta_1).e_1(t_i^-), \quad i \in N \end{cases} \tag{32}$$

where $\beta_1, \beta_2, \beta_3$ are the positive real observer gains to be given later by Theorem1 and $Z(e, y(t_i))$ is the term given in section 2.3 by

$$Z(e,y) = D(q) = [I - M_e^{-1}.J^T.(J.M_e^{-1}.J^T)^{-1}J] \tag{33}$$

To demonstrate the asymptotically convergence of the error dynamics to zero (i.e. the convergence of $(e_1(t), e_2(t))$ to $(0,0)$ as $t \to \infty$), we define here a definite positive Lyapunov function that regroups the dynamics of $e_1(t)$ and $e_2(t)$. The proposed function is given by:

$$V(t) = \frac{1}{2} r(t)^T.r(t) \tag{34}$$

The objective is to find sufficient conditions on $\beta_1, \beta_2, \beta_3$ so that the time derivative of V(t) is negative definite which make the Lyapunov function V(t) continually decreasing.

For $t \in]t_i, t_{i+1}[$, the time derivate of (12) gives:

$$\dot{V}(t) = r(t)^T.\dot{r}(t) \tag{35}$$

When substituting all variables by their corresponding expressions into (35), using physical properties mentioned in section 2. C, and under assumption 1 (we replace $\operatorname{sgn}(e_1(t))$ by $\operatorname{sgn}(r(t))$), we leads finally to the following expression

$$\begin{aligned} \dot{V} &= -r^T.(M(y))^{-1}.[C(y,\hat{v}) + C(y,v)].r \\ &\quad + \alpha.r^T.(M(y))^{-1}.[C(y,\hat{v}) + C(y,v)].e_1 \\ &\quad + r^T.(M(y))^{-1}.d(t,y,v) - \beta_2.|r^T|.|e_1| \\ &\quad - \beta_3.|r^T| - r^T.\beta_1.r - \beta_1^2.|r^T|.|e_1| \end{aligned} \tag{36}$$

(For more details, see our analysis convergence in our work given in Mchiri[17]).

The following assumptions are required for the rest of our analysis.

Assumption2. The initial conditions of the state vector of the mechanical system (12) $[q^T(t_0) \quad \dot{q}^T(t_0)]^T$ and the control force *u(t)* are chosen so that the position and the velocity vector are bounded functions of time, i.e., there exist two constants $k_q, k_v > 0$ such that $\|q(t)\| < k_q$ and $\|\dot{q}(t)\| < k_v$, for all times $t \geq t_0$. Note that, when $t = t_i$, which corresponds to an impact, t is substituted by t_i^+ and t_i^-. By this assumption, the continuity of M(.) and C(.,.), the invertibility of M(.), and the linearity of $C(q,\dot{\hat{q}})$ with respect to $\dot{\hat{q}}$, there exist two constants $k_1, k_2 > 0$ such that

$$\|M^{-1}(y(t)).C(y(t), x_2(t))\| \leq k_1 \tag{37}$$

$$\|M^{-1}(y(t)).C(y(t), \hat{x}_2(t))\| \leq k_2.k_v + k_2.\|\tilde{v}(t)\| \tag{38}$$

where $\tilde{v}(t) = \|\hat{v}(t) - v(t)\| = \|\hat{x}_2(t) - x_2(t)\|$ and $\|.\|$ denotes the matrix norm induced by the Cartesian norm for vectors.

Assumption3. If the continuous part of system (12) is affected by an external perturbing term $d(t,y,v)$, that representing uncertainties, then this term is assumed to be bounded by a positive constant. Since the position vector is a bounded function of time and by the continuity of *M(.)*, we assume then, that the term $M^{-1}(y).\Delta(t,y,x_2)$ is bounded by a positive constant μ, i.e.

$$\|M^{-1}(y).d(t,y,v)\| < \mu \tag{39}$$

So, under assumptions 1, 2, and 3, we can now upper bound the right-hand side of (36) as follows:

$$\begin{aligned} \dot{V} &\leq +\|r\|^2.(k_1 + k_2.k_v + k_2.\|\tilde{v}(t)\|) \\ &\quad + \alpha.(k_1 + k_2.k_v + k_2.\|\tilde{v}(t)\|).\|r\|.\|e_1\| \\ &\quad + \mu.\|r\| - \beta_2.\|r\|.\|e_1\| \\ &\quad - \beta_3.\|r\| - \beta_1.\|r\|^2 - \beta_1^2.\|r\|.\|e_1\| \end{aligned} \tag{40}$$

Let $\sigma = k_1 + k_2.k_v$. So, we have:

$$\begin{aligned} \dot{V} &\leq -\|r\|^2.[\beta_1 - (\sigma + k_2.\|\tilde{v}(t)\|)] \\ &\quad -\|r\|.\|e_1\|.[\beta_2 - (\alpha.(\sigma + k_2.\|\tilde{v}(t)\|) - \beta_1^2)] \\ &\quad -\|r\|.(\beta_3 - \mu) \end{aligned} \tag{41}$$

Finally, if the following conditions are satisfied

$$\begin{cases} \beta_1 > \sigma \\ \beta_2 > \alpha.\sigma - \beta_1^2 \\ \beta_3 > \mu \end{cases} \tag{42}$$

we can easily obtain a negative semi-definite function in a neighbourhood of $\tilde{v} = 0$, from which we can guarantee an attractive and invariant sliding surface. So, for each $t \in]t_i, t_{i+1}[$ and under conditions given by system (42), V(t) is a positive-definite Lyapunov function whose time derivative is negative semi-definite. By LaSalle's invariance theorem, we have $r(t) \to 0$ as $t \to \infty$. Under assumption 1 and after taking an appropriate choice of the scalar α, the signal $r(t) = 0$ means $e_1(t) = e_2(t) = 0$: the local asymptotic velocity observation is then guaranteed for each $t \in]t_i, t_{i+1}[$.

At each impact time t_i, the Lyapunov function V is also decreasing. To demonstrate this property, the following assumption is needed for our analysis.

Assumption4. The term representing the velocity component of the impact map satisfies

$$\|Z(e,q)\| \le 1, \quad \forall e \in [0,1], \quad \forall q \in \Re^n, \qquad (43)$$

For model (12), since the impact does not change the position configuration (positions remain continuous), we have $q^+ = q^- = q$. Consequently, the post-impact and the pre-impact errors are equals. Let $e_1(t_i^+) = e_1(t_i^-) = e_1(t_i)$. At each impact time t_i, jumps are imposed in the estimated velocity vector

$$\hat{v}(t_i^+) = Z(e, y(t_i)).\hat{v}(t_i^-) \qquad (44)$$

Then, using (9) and (44), the jumps at the impact times of the velocity estimation error are given by

$$\tilde{v}(t_i^+) = Z(e, y(t_i)).\tilde{v}(t_i^-) \qquad (45)$$

So, under assumption2, and using (34) and (45), we have

$$\begin{aligned} V(t_i^+) &\le \frac{1}{2}[(e_2^T(t_i^-).e_2(t_i^-) + \alpha^2.e_1(t_i))^2 \\ &\quad + 2.\alpha.|e_2(t_i^-)|.|e_1(t_i)|] \\ &\le \frac{1}{2}[(e_2(t_i^-))^2 + \alpha^2.e_1(t_i))^2 \\ &\quad + 2.\alpha.|e_2(t_i^-)|.|e_1(t_i)|] \qquad (46) \\ &\le \frac{1}{2}[(e_2(t_i^-) + \alpha.e_1(t_i))^2] \\ &\le \frac{1}{2}(r(t_i^-))^2 \\ &\le V(t_i^-) \end{aligned}$$

So at each impact time t_i, we have also a decreasing Lyapunov function V.

Finally, provided the conditions given by system (42), the observer (32) ensures a finite time local asymptotically convergence of both continuous and discrete estimated states to real states of system (12), i.e. $(\hat{x}_1, \hat{x}_2) \to (x_1, x_2)$ in a finite time.

The overall system taking into account the control law and the observer design is finally given by

$$\begin{cases} \dot{x} = f(x) + g(x).\hat{u}_1 & , \quad x^- \notin S \\ \dot{\hat{x}} = \hat{f}(\hat{x}) + g(\hat{x}).\hat{u}_1 - L_1(\hat{y}_M - y_M) & \\ \quad - L_2.\mathrm{sign}(\hat{y}_M - y_M), & \\ \hat{u}_1 = h(\hat{x}) & \\ y_M = x_1 & \\ \hat{y}_M = \hat{x}_1 & \\ x^+ = \Delta'(x^-) & , \quad x^- \in S \\ \hat{x}^+ = \hat{\Delta}'(\hat{x}^-) & \end{cases} \qquad (47)$$

where $L_1 = [\alpha + \beta_1 \quad \beta_2]$ and $L_2 = [0 \quad \beta_3]$. Let $e = \hat{x} - x$ be the estimation error vector. So (47) can be rewritten as

$$\begin{cases} \dot{x} = f(x) + g(x).\hat{u}_a(x+e) & , \quad x^- \notin S \\ \dot{e} = \hat{F}(x,e) & \\ y_M = x_1 & \\ \hat{y}_M = x_1 + e_1 & \\ x^+ = \Delta'(x^-) & , \quad x^- \in S \\ e^+ = \hat{\Delta}'(x^-, e^-) & \end{cases} \qquad (48)$$

where

$$\begin{aligned} \hat{F}(x,e) = \hat{f}(x+e) - f(x) + (g(x+e) - g(x))\hat{u}_a(x+e) \\ - L_1(\hat{y}_M - y_M) - L_2\mathrm{sgn}(\hat{y}_M - y_M), \end{aligned}$$

and

$$\Delta'(x^-, e^-) = \Delta'(x^- + e^-) - \Delta'(x^-).$$

V. SIMULATION RESULTS

For simulation purposes, parameters are taken as follows: the length of a leg (*r = 1 m*), the mass of a leg (*m = 5kg*), the mass of hips (*Mh=10 kg*), the mass of the torso (*Mt =10 kg*), the distance between hips and torso (*L=0.5 m*) and *g=9.8* the gravity acceleration. The control law parameters are given by $K_0 = \mathrm{diag}(10,10)$, $K_1 = \mathrm{diag}(15,15)$, and $y_{1d} = y_{2d} = \cos(5t) + \sin(t)$. The observer gains are given by $\alpha = 3{,}5.10^2, \beta_1 = 686, \quad \beta_2 = 13470, \beta_3 = 2$. Simulation results given by figures 2 and 3 show the efficiency of our proposed method applied to the planar biped robot of figure 1. It can be clearly seen, from figures 2 and 3 that the proposed observer provides an excellent estimation of the robot position variables (stance leg and swing leg). In each figure, are shown both the real and estimated continuous position variables and the finite-time convergence of the position error estimation to zero. Figure 4 shows the finite time velocity estimation error of each link of the biped robot (the two legs and the torso). It can be seen from this figure that our observer provides a good estimation of the velocity variables in a finite time. Figures 5 and 6 display respectively the phase portrait of the stance leg and the swing leg of the biped robot. It can be clearly seen, from these figures, that the controller (18) ensures stable cycles. The control signals, provided by the proposed controller, are strongly based on the estimated variables and shown in figure 7.

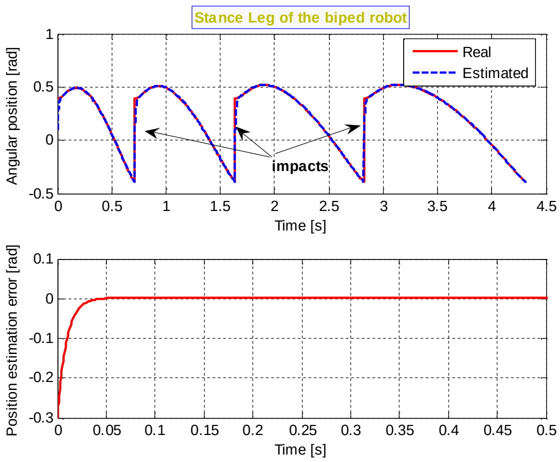

Fig. 2 Position variables of the stance leg of the biped robot : (a) Continuous position variables (real and estimated); (b) Finite-time convergence of the position error estimation to zero.

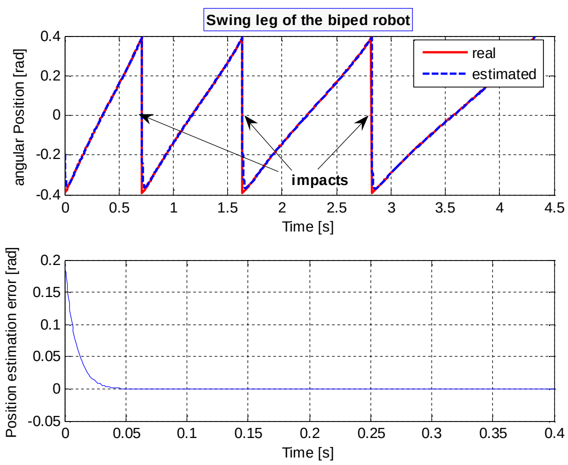

Fig.3 Position variables of the swing leg of the biped robot: (a) Continuous position variables (real and estimated); (b) Finite-time convergence of the position error estimation to zero.

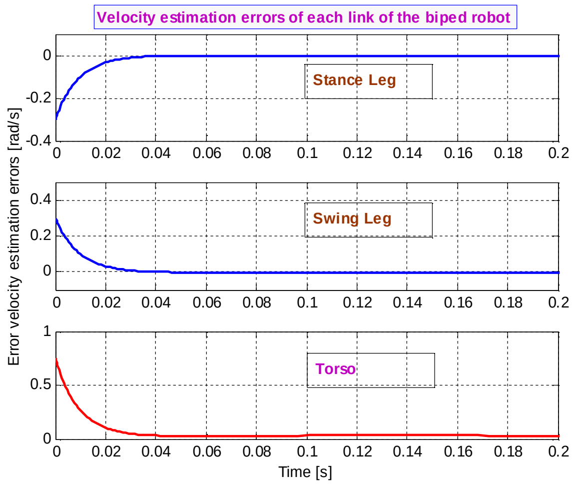

Fig.4 Finite-time convergence of the velocity estimation error of each link of the biped robot: (a) stance leg; (b) swing leg; (c) Torso.

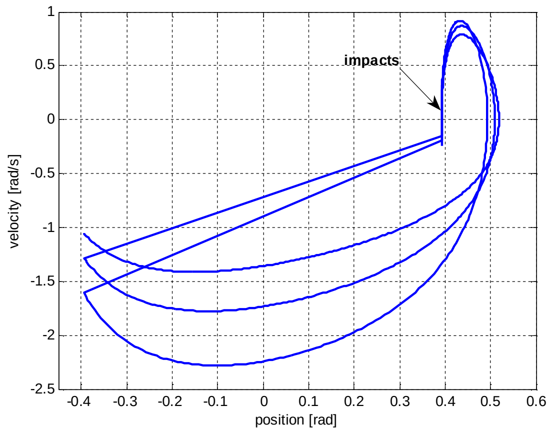

Fig. 5 Stable walking cycle : Phase portrait of the stance leg of the planar robot (angular position in radians versus velocity in radians/seconds)

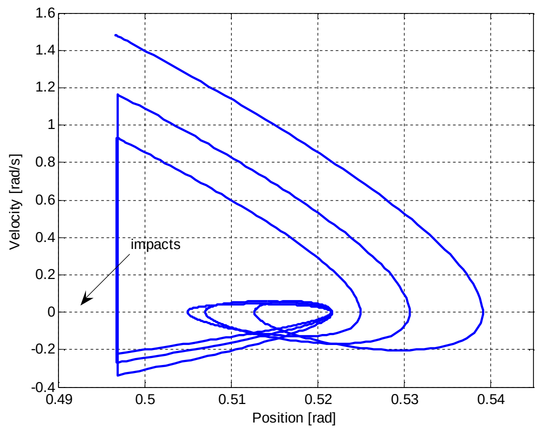

Fig. 6 Stable walking cycle: Phase portrait of the swing leg of the planar robot (angular position in radians versus velocity in radians/seconds).

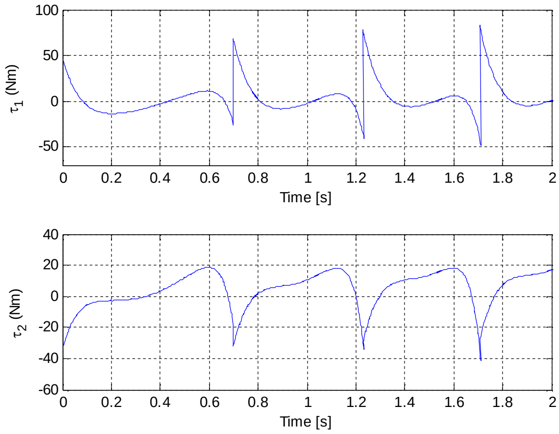

Fig. 7 Applied torques based on estimated variables.

VI. Conclusions

In this paper, we have proposed a partial feedback linearization technique for controlling an under-actuated walking biped robot. After the formulation of the hybrid model of the system under consideration, we have shown that the vector of actuated configurations can be completely linearized and decoupled from the vector of non-actuated configurations. Then, since the velocity signals are not available, the control law was coupled with our proposed hybrid observer in order to estimate the velocity vector of the mechanical system. Simulation results applied to the three-link biped robot show that the control strategy, based on estimated variables, induces exponentially stable walking locomotion.

Appendix

Appendix1: Matrices details of the biped robot

This Appendix contains the definition of all matrices of the biped model for continuous motion part

Inertia Matrix : M(q)

$$M(q)=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \text{ where}$$

$$m_{11}=(\frac{5}{4}m+M_H+M_T).r^2$$
$$m_{12}=m_{21}=-\frac{1}{2}m.r^2.\cos(\theta_1-\theta_2)$$
$$m_{13}=m_{31}=M_T.r.l.\cos(\theta_1-\theta_3)$$
$$m_{22}=\frac{1}{4}m.r^2$$
$$m_{23}=m_{32}=0, \quad m_{33}=M_T.l^2$$

Coriolis and centrifugal forces matrix: C

$$C(q,\dot{q})=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \text{ where}$$

$$c_{11}=c_{22}=c_{33}=c_{23}=c_{32}=0$$
$$c_{12}=-\frac{1}{2}m.r^2.\sin(\theta_1-\theta_2).\dot{\theta}_2$$
$$c_{21}=\frac{1}{2}m.r^2.\sin(\theta_1-\theta_2).\dot{\theta}_1$$
$$c_{13}=M_T.r.l.\sin(\theta_1-\theta_3).\dot{\theta}_3$$
$$c_{31}=-M_T.r.l.\sin(\theta_1-\theta_3).\dot{\theta}_1$$

Vector of gravitational forces : G(q)

$$G(q)=\begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix}$$

where:

$$g_{11}=-\frac{1}{2}.g.(2M_H+3.m+2M_T).r.\sin(\theta_1)$$
$$g_{21}=\frac{1}{2}.g.m.r.\sin(\theta_2)$$
$$g_{31}=-.g.M_T l.\sin(\theta_3)$$

Matrix of the effects of actuators on the generalized coordinates B(q)

$$B=\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Appendix2: Matrices corresponding to the impact model

$$M_e(q_e).(\dot{q}_e^{\,+} - \dot{q}_e^{\,-}) = F_{ext}$$

where

$$M_e(1,1) = \frac{1}{4}(5.m + 4M_H + 4.M_T).r^2$$

$$M_e(2,2) = \frac{1}{4}m.r^2 ,$$

$$M_e(1,2) = M_e(2,1) = -\frac{1}{2}m.r^2 \cos(-\theta_1 + \theta_2) ,$$

$$M_e(2,3) = M_e(3,2) = 0$$

$$M_e(1,3) = M_e(3,1) = M_T.r.l.\cos(\theta_1 - \theta_3)$$

$$M_e(2,4) = M_e(4,2) = -\frac{1}{2}m.r.\cos(\theta_2) ,$$

$$M_e(1,4) = M_e(4,1) = \frac{1}{2}(3.m + 2M_H + 2M_T).r.\cos(\theta_1)$$

$$M_e(1,5) = M_e(5,1) = -\frac{1}{2}(3.m + 2M_H + 2M_T).r.\sin(\theta_1)$$

$$M_e(2,5) = M_e(5,2) = \frac{1}{2}m.r.\sin(\theta_2) \quad ,$$

$$M_e(3,3) = M_T.l^2 \quad ,$$

$$M_e(3,4) = M_e(4,3) = M_T.l.\cos(\theta_3) \quad ,$$

$$M_e(3,5) = M_e(5,3) = -M_T.l.\sin(\theta_3) ,$$

$$M_e(4,5) = M_e(5,4) = 0 \quad ,$$

$$M_e(4,4) = M_e(5,5) = 2m + M_H + M_T$$

Appendix3: partitioned matrices used in control law

This Appendix contains the partitioned matrices of the biped model utilized for the development of the feedback controller

$$M_{11} = M_T.l^2$$

$$M_{12} = \begin{bmatrix} M_T.r.l.\cos(\theta_1 - \theta_3) & 0 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} M_T.r.l.\cos(\theta_1 - \theta_3) \\ 0 \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} (\frac{5}{4}m + M_H + M_T).r^2 & -\frac{1}{2}m.r^2.\cos(\theta_1 - \theta_2) \\ -\frac{1}{2}m.r^2.\cos(\theta_1 - \theta_2) & \frac{1}{4}m.r^2 \end{bmatrix}$$

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