# Free Vibration Characteristics of Edge Cracked Functionally Graded Beams by Using Finite Element Method

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***Abstract*—This paper presents free vibration analysis of an edge cracked functionally graded cantilever beam. The differential equations of motion are obtained by using Hamilton's principle. The considered problem is investigated within the Euler-Bernoulli beam theory by using finite element method. The cracked beam is modeled as an assembly of two sub-beams connected through a massless elastic rotational spring. Material properties of the beam change in the thickness direction according to exponential distributions. In order to establish the accuracy of the present formulation and results, the natural frequencies are obtained, and compared with the published results available in the literature. Good agreement is observed. In the study, the effects of the location of crack, the depth of the crack and different material distributions on the natural frequencies and the mode shapes of the cracked functionally graded beams are investigated in detail.**

***Keywords*—Open edge crack, Free vibration, Functionally graded materials, Finite element analysis**

## I. INTRODUCTION

The effect of crack on the vibration behavior is an important topic in structural safety assessment and has received increasing research efforts.

Functionally graded materials (FGMs), a novel class of composites whose composition varies continuously as a function of position along thickness of a structure to achieve a required function. Functionally graded structures have been an area of intensive research overthe last decade.

In the literature, the free vibration and dynamic behavior of homogeneous cracked beams have been extensively studied [1-27]. In recent years, the dynamic behavior of cracked FGM beams has been a topic of active research. Sridhar et al. [28] developed an effective pseudo-spectral finite element method for wave propagation analysis in anisotropic and inhomogeneous structures with or without vertical and horizontal cracks. Briman and Byrd [29] studied the effect of damage on free and forced vibrations of a functionally graded cantilever beam. Yang et al. [30] investigated an analytical study on the free and forced vibration of inhomogeneous Euler–Bernoulli beams containing open edge cracks that the beam is subjected to an axial compressive force and a concentrated transverse load moving along the longitudinal direction. Yang and Chen [31] investigated free vibration and buckling analysis of FGM beams with edge cracks by using Bernoulli–Euler beam theory and the rotational spring model. Free vibration and elastic buckling of beams made of FGM containing open edge cracks are studied within Timoshenko beam theory by Ke et al. [32]. Yu and Chu [33] studied the transverse vibration characteristics of a cracked FGM beam by using the p-version of finite element method. Ke et al. [34] investigated the post-buckling analysis of FGM beams with an open edge crack based on Timoshenko beam theory and von Kármán nonlinear kinematics by using Ritz method. Matbuly et al. [35] studied the free vibration analysis of a cracked FGM beam resting on a Winkler–Pasternak foundation by using differential quadrature method. Ferezqi et al. [36] studied an analytical investigation of the free vibrations of a cracked Timoshenko beam made up of FGM. Yan et al. [37] studied dynamic response of FGM beams with an open edge crack resting on an elastic foundation subjected to a transverse load moving at a constant speed. Akbaş [38] investigated static analysis of an edge cracked FGM beam resting on Winkler foundation by using finite element method. Yan et al. [39] investigated the nonlinear flexural dynamic behavior of a clamped Timoshenko beam made of FGM with an open edge crack under an axial parametric excitation which is a combination of a static compressive force and a harmonic excitation force based on Timoshenko beam theory and von Kármán nonlinear kinematics. Wei et al. [40] studied the free vibration of cracked FGM beams with axial loading, rotary inertia and shear deformation by using an analytical method. In a recent study, Akbaş [41] investigated geometrically nonlinear analysis of edge cracked Timoshenko beam by using Total Langragian finite element method.

In the literature studies, the effect of the parameter of FGM property distribution on response of vibration and crack is limited in numerical results. To obtain more realistic answers and understand to edge cracked FGM beam, more the parameter of FGM property distribution must be used in numerical results. Hence, a lot of the parameters of FGM property distribution are used in this study.

The differential equations of motion are obtained by using Hamilton's principle. The considered problem is investigated within the Euler-Bernoulli beam theory by using finite element method. The cracked beam is modeled as an assembly of two sub-beams connected through a massless elastic rotational spring. Material properties of the beam change in the thickness direction according to exponential distributions. In the study, the effects of the location of crack, the depth of the crack and different material distributions on the natural frequencies and the mode shapes of the functionally graded beams are investigated in detail. Also, some of the present results are compared with the previously published results to establish the validity of the present formulation.

## II. THEORY AND FORMULATIONS

Consider a cantilever FGM beam of length *L*, width *b*, thickness *h*, containing an edge crack of depth *a* located at a distance $L_1$ from the left end as shown in Figure 1. It is assumed that the crack is perpendicular to beam surface and always remains open.

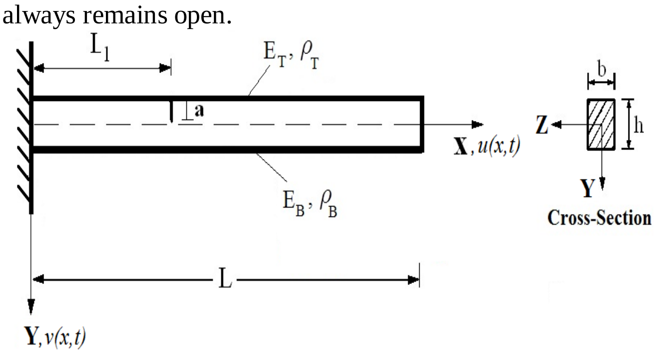


**Figure 1.** A cantilever FGM beam with an open edge crack and cross-section.

In this study, Young's modulus $E(Y)$ and mass density $\rho(Y)$ vary continuously in the thickness direction ($Y$ axis) according to exponential distributions as follows;

$$E(Y) = E_0 e^{\beta Y}, \quad \rho(Y) = \rho_0 e^{\beta Y} \tag{1}$$

### A. Governing equation of free vibration of intact FGM beams

Acoording to the coordinate system *(X,Y,Z)* shown in figure 1, based on Euler-Bernoulli beam theory, the axial and the transverse displacement field are expressed as

$$u(X,Y,t) = u_0(X,t) - Y\frac{\partial v_0(X,t)}{\partial X} \tag{2}$$

$$v(X,Y,t) = v_0(X,t) \tag{3}$$

Where $u_0$ and $v_0$ are the axial and the transverse displacements in the mid-plane, $t$ indicates time. Using Eq. (2) and (3), the linear strain- displacement relation can be obtained:

$$\varepsilon_{xx} = \frac{\partial u}{\partial X} = \frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial^2 v_0(X,t)}{\partial X^2} \tag{4}$$

According to Hooke's law, constitutive equations of the FGM beam are as follows:

$$\sigma_{xx} = E(Y)\,\varepsilon_{xx} = E(Y)\left[\frac{\partial u_0(X,t)}{\partial X} - Y\frac{\partial^2 v_0(X,t)}{\partial X^2}\right] \tag{5}$$

Where $\sigma_{xx}$ and $\varepsilon_{xx}$ are normal stresses and normal strains in the X direction, respectively. Based on Euler-Bernoulli beam theory, the elastic strain energy (*V*) and kinetic energy (*T*) of the FGM beam are expressed as

$$V = \frac{1}{2}\int_0^L \int_A \sigma_{xx}\varepsilon_{xx}\, dA dX \tag{6}$$

$$T = \frac{1}{2}\int_0^L \int_A \rho(Y)\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] dA\, dX \tag{7}$$

With applying Hamilton's principle, the differential equations of motion are obtained as follows:

$$A_{XX}\frac{\partial^2 u_0}{\partial X^2} - B_{XX}\frac{\partial^2 v_0}{\partial X^2} = I_1\frac{\partial^2 u_0}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial v_0}{\partial X}\right) \tag{8}$$

$$B_{XX}\frac{\partial^2}{\partial X^2}\left(\frac{\partial u_0}{\partial X}\right) - D_{XX}\frac{\partial^2}{\partial X^2}\left(\frac{\partial^2 v_0}{\partial X^2}\right) = -I_3\frac{\partial^2}{\partial t^2}\left(\frac{\partial v_0}{\partial X}\right) + I_1\frac{\partial^2 v_0}{\partial t^2} \tag{9}$$

The stiffness components are defined as

$$(A_{XX}, B_{XX}, D_{XX}) = \int_A E(Y)(1, Y, Y^2)\, dA \tag{10}$$

$$(I_1, I_2, I_3) = \int_A \rho(Y)(1, Y, Y^2)\, dA \tag{11}$$

### B. Finite Element Formulations

The displacement field of the finite element shown is expressed in terms of nodal displacements as follows:

$$u_0^e(X,t) = \varphi_1^U(X)\, u_1(t) + \varphi_2^U(X)\, u_2(t) \tag{12}$$

$$v_0^{(e)}(X,t)=\varphi_1^{(V)}(X)\,v_1(t)+\varphi_2^{(V)}(X)\,\theta_1(t)+\varphi_3^{(V)}(X)v_2(t)+\varphi_4^{(V)}(X)\,\theta_2(t) \tag{13}$$

where $u_i$ , $v_i$ and $\theta_i$ are axial displacements, transverse displacements and slopes at the two end nodes of the beam element, respectively. $\varphi_i^{(U)}$ and $\varphi_i^{(V)}$ are interpolation functions for axial and transverse degrees of freedom, respectively, which are given in Appendix. Two-node beam element shown in figure 2.

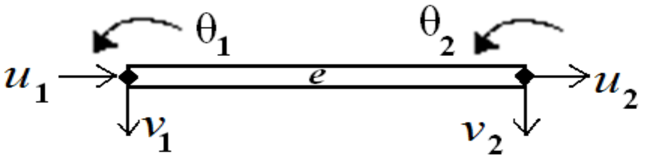


**Figure 2.** A two-node beam element

With using the standard procedure of the Galerkin finite element method, the stiffness matrix and the mass matrix are obtained according to Eqs. (8) and (9). The equation of motion as follows:

$$[K]\{q\}+[M]\{\ddot{q}\}=0 \tag{14}$$

where $[K]$ is the stiffness matrix and $[M]$ is the mass matrix. $\{q\}$ is nodal displacement vector which as follows

$$\{q\}=\{u,v,\theta\}^T \tag{15}$$

The components of the stiffness matrix $[K]$:

$$[K]=\begin{bmatrix}[K^A] & [K^B]\\ [K^B]^T & [K^D]\end{bmatrix}, \tag{16a}$$

Where

$$[K^A]=\int_0^{L_e} A_{XX}\left[\frac{d\varphi^{(U)}}{dX}\right]^T\left[\frac{d\varphi^{(U)}}{dX}\right]dX\,, \tag{16b}$$

$$[K^B]=-\int_0^{L_e} B_{XX}\left[\frac{d^2\varphi^{(V)}}{dX^2}\right]^T\left[\frac{d\varphi^{(U)}}{dX}\right]dX\,, \tag{16c}$$

$$[K^D]=\int_0^{L_e} D_{XX}\left[\frac{d^2\varphi^{(V)}}{dX^2}\right]^T\left[\frac{d^2\varphi^{(V)}}{dX^2}\right]dX\,, \tag{16d}$$

The mass matrix $M$ can be expressed as a sum of four sub-matrices as shown below:

$$[M]=[M_U]+[M_V]+[M_\theta]+[M_{U\theta}] \tag{17}$$

Where $[M_U]$, $[M_V]$ and $[M_\theta]$ are the contribution of *u*, *v* and $\theta$ degree of freedom to the mass matrix, $[M_{U\theta}]$ is coupling mass matrix due to coupling between $u$ and $\theta$ . Explicit forms of $[M]$ are given in Appendix.

### *C. Crack Modeling*

The cracked beam is modeled as an assembly of two sub-beams connected through a massless elastic rotational spring shown in figure 3.

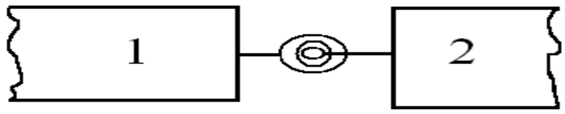


**Figure 3.** Rotational spring model.

The bending stiffness of the cracked section $k_T$ is related to the flexibility G by

$$k_T=\frac{1}{G} \tag{18}$$

Cracked section's flexibility G can be derived from Broek's approximation [42]:

$$\frac{(1-\nu^2)K_I^2}{E(a)}=\frac{M^2}{2}\frac{dG}{da} \tag{19}$$

where $M$ is the bending moment at the cracked section, $K_I$ is the stress intensity factor (SIF) under mode I bending load and is a function of the geometry, the loading, and the material properties as well. $\nu$ indicates Poisson's ratio which is taken to be constant since its influence on the stress intensity factors is quite limited [43]. For an FGM strip with an open edge crack under bending, the analytical solution and the expression of SIF is given Yu And Chu [33] that obtained from the data given by Erdogan and Wu [43] through Lagrange interpolation technique:

$$K_I=\frac{6M}{bh^2}\sqrt{\pi a}\,F(E_R,a/h) \tag{20}$$

Where, $a$ is crack of depth, $E_R$ is the ratio of Young's modulus of bottom and top surfaces of the beam ($E_B/E_T$)

and $F$ is an unknown function of two independent variables. The function $F$ is can be expressed as follows [33];

$$F(E_R, a/h) = \frac{p_1 + p_2 \ln(E_R) + p_3[\ln(E_R)]^2 + p_4[\ln(E_R)]^3 + p_5(a/h) + p_6(a/h)^2}{1 + p_7 \ln(E_R) + p_8[\ln(E_R)]^2 + p_9(a/h) + p_{10}(a/h)^2 + p_{11}(a/h)^3} \quad (21)$$

Where the coefficients $p_1$, $p_2$, …, $p_{10}$, $p_{11}$ =1.1732, -0.3539, 0.0289, -0.0061, 0.6625, 3.072, -0.0014, -0.0017, 1.9917, -0.3496, -3.0982 are given in Yu And Chu [33] that are determined by fitting Eq. (20) based on the least square method to the numerical values of the SIF for specific material gradients and normalized crack size given by Erdogan and Wu [43].

The spring connects the adjacent left and right elements and couples the slopes of the two FGM beam elements at the crack location. In the massless spring model, the compatibility conditions enforce the continuities of the axial displacement, transverse deflection, axial force and bending moment across the crack at the cracked section ( $X = L_1$ ), that is,

$$u_1 = u_2,\ v_1 = v_2,\ N_1 = N_2,\ M_1 = M_2 \quad (22)$$

The discontinuity in the slope is as follows:

$$k_T\left(\frac{dv_1}{dX} - \frac{dv_2}{dX}\right) = k_T(\theta_1 - \theta_2) = M_1 \quad (23)$$

Based on the massless spring model, the stiffness matrix of the cracked section as follows:

$$[K]_{(Cr)} = \begin{bmatrix} 1/G & -1/G \\ -1/G & 1/G \end{bmatrix} = \begin{bmatrix} k_T & -k_T \\ -k_T & k_T \end{bmatrix} \quad (24)$$

The stiffness matrix of the cracked section is written according to the displacement vector:

$$\{q\}_{(Cr)} = \{\theta_1, \theta_2\}^T \quad (25)$$

Where $\theta_1$ and $\theta_2$ are the angles of the cracked section. With adding crack model, the equations of motion for the finite element and by use of usual assemblage procedure the following system of equations of motion for the whole system can be obtained as follows:

$$([K] + [K]_{(Cr)})\{q\} + [M]\{\ddot{q}\} = 0 \quad (26)$$

If the global nodal displacement vector $\{q\}$ is assumed to be harmonic in time with circular frequency $\omega$ , i.e $\{q\} = \{\bar{q}\}e^{i\omega t}$ becomes, after imposing the appropriate end conditions, an eigenvalue problem of the form:

$$([K] + [K]_{(Cr)} - \omega^2[M])\{\bar{q}\} = 0 \quad (27)$$

Where $\{\bar{q}\}$ is a vector of displacement amplitudes of the vibration. The dimensionless quantities can be expressed as

$$\bar{\omega} = \frac{\omega}{\sqrt{D_0 / I_{10}}},\ \bar{X} = \frac{X}{L}, \quad (28)$$
$$\bar{Y} = \frac{Y}{h},\ E_R = \frac{E_B}{E_T},\ \rho_R = \frac{\rho_B}{\rho_T}$$

Where $\bar{\omega}$ is the dimensionless frequency, $E_R$ is the ratio of Young's modulus of bottom and top surfaces of the beam, $\rho_R$ is the ratio of mass density of bottom and top surfaces of the beam. $D_0$ and $I_{10}$ indicate the value of $D_{xx}$ and $I_1$ of an isotropic homogeneous beam.

## III. NUMERICAL RESULTS

In the numerical examples, the natural frequencies and the mode shapes of the beams are calculated and presented in figures for different the location of crack,s the depth of the cracks, material distributions. The beam considered in numerical examples is made of Aluminum ( $E = 70\,GPa,\ \nu = 0.33,\ \rho = 2780\ kg/m^3$ ) which the material constants change exponentially as in Eq. (1). The top surface of the FGM beam is Aluminum. In the numerical integrations, five-point Gauss integration rule is used. Unless otherwise stated, it is assumed that the width of the beam is $b = 0.1m$, height of the beam is $h = 0.1m$ and length of the beam is $L = 30h$ in the numerical results. In the numerical calculations, the number of finite elements is taken as $n$ = 100.

In order to establish the accuracy of the present formulation and the computer program developed by the author, the results obtained from the present study are compared with the available results in the literature. For this purpose, the fundamental frequency of a cantilever isotropic homogeneous beam with an open edge are calculated for various the location of crack ( $L_1/L$ ) for $L = 0.2\,m$ , $h = 0.0078m$ , $a/h = 0.2$ , $E = 216\,GPa$, $\rho = 7850\ kg/m^3$ , $\nu = 0.28$, and compared with those of Kısa et al. [3] and Ke et al.[32]. As seen from Table 1, the present results are close to the results of Kısa et al. [3] and Ke et al.[32].

TABLE I

Fundamental frequency $\omega_1$ of a cantilever isotropic homogenous beam with an edge crack

| | $L_1/L=0.2$ | $L_1/L=0.4$ | $L_1/L=0.6$ | Intact Beam |
|---|---|---|---|---|
| Present | 1021.6 | 1031.2 | 1036.2 | 1037.09 |
| Kısa et al. [3] | 1020.0137 | 1030.095 | 1035.284 | 1037.0189 |
| Ke et al.[32] | 1020.098 | 1029.853 | 1034.932 | 1037.0106 |

To further verify the present results, the dimensionless fundamental frequency of a FGM cantilever beam without crack are calculated for various $E_R$ ratio for $L/h=20$ compared with those of Yang and Chen. [31] and Ke et al.[32]. As seen from Table 2, the present results are in good agreement with that the results of Yang and Chen. [31] and Ke et al.[32].

TABLE II

Comparison of the dimensionless fundamental frequency $\bar{\omega}_1$ of intact FGM beams

| $E_R$ | Present | Yang and Chen [31] | Ke et al.[32] |
|---|---|---|---|
| 0,2 | 0.8283 | 0.83 | 0.8235 |
| 1 | 0.8786 | 0.88 | 0.8752 |
| 5 | 0.8283 | 0.83 | 0.8235 |

In figure 4, the effect of Young's modulus ratio $E_R$ on the dimensionless fundamental frequency $\bar{\omega}_1$ of edge cracked FGM beams ($L_1/L=0.05$) is shown for different the crack depth ratios $a/h$.

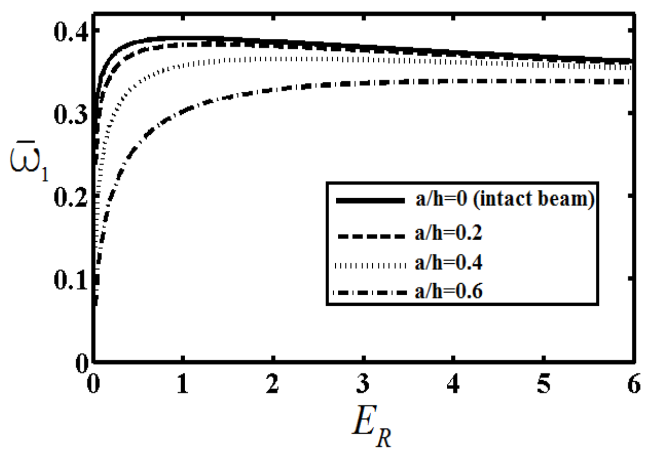


**Figure 4** The effect of Young's modulus ratio $E_R$ on the dimensionless fundamental frequency $\bar{\omega}_1$ for different the crack depth ratios $a/h$.

It is seen from figure 4 that with increase in the crack depth, the fundamental frequency decreases, as expected. It is an interesting point for Fig. 4 that there is an exponential symmetry with respect to the vertical line at $E_R=1$ for intact FGM beam ($a/h=0$) that the effective material properties are expressed as an exponential function (Eq. 1). It is seen from figure 4 that the dimensionless fundamental frequencies of Young's modulus ratios $E_R=2$ and 0.5, $E_R=5$ and 0.2, $E_R=8$ and 0.25… etc. are same for intact FGM beam ($a/h=0$). Because their values of $I_{10}/A_{10}$ are almost identical. It is pointed before that $I_{10}$ and $A_{10}$ indicate the value of $I_1$ and $A_{XX}$ of an isotropic homogeneous beam, respectively. There is no an exponential symmetry with respect to the vertical line at $E_R=1$ for cracked FGM beam ($a/h\neq 0$). Hence, the dimensionless fundamental frequencies of Young's modulus ratios $E_R=2$ and 0.5, $E_R=5$ and 0.2, $E_R=8$ and 0.25… etc. are not same for cracked FGM beam ($a/h\neq 0$). With increase Young's modulus ratio $E_R$, the fundamental frequency increases for cracked beam. It is observed from figure 4 that there are significant differences of the mechanical behaviour for the cracked and intact FGM beams.

In figure 5, the effect of Young's modulus ratio $E_R$ on the dimensionless fundamental frequency $\bar{\omega}_1$ of edge cracked FGM beams ($a/h=0.6$) is shown for different the crack locations ($L_1/L$) from the left end.

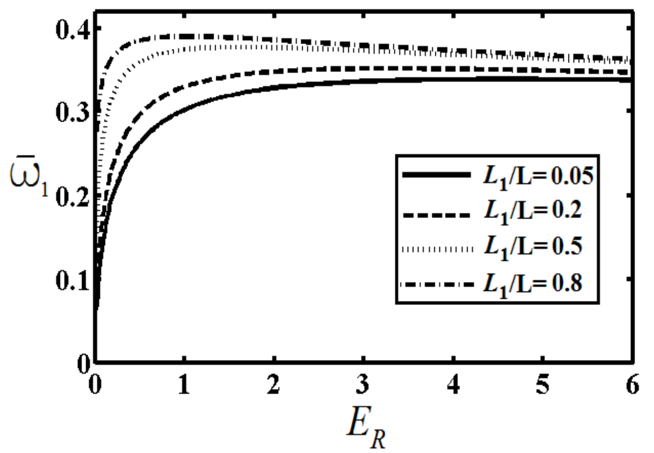


**Figure 5** The effect of Young's modulus ratio $E_R$ on the dimensionless fundamental frequency $\bar{\omega}_1$ for various the crack locations ($L_1/L$)

It is obviously observed from figure 5 that with the crack locations get closer to the left end, the fundamental frequency decreases. Because the fixed support has the most stress, strain and the rigidity point in the beam. Hence, the crack gets closer to fixed support, the beam gets more flexible. The crack gets closer to free end, the effect of crack on the fundamental frequency for FGM beam decrease. An another result of the figure 5 that with the increase Young's modulus ratio $E_R$, the difference between with cracked and intact of FGM beam decreases seriously. It shows that with the suitable choice of $E_R$, the negative effects of the crack can be reduced. Also, it is seen figure 5 that with the crack gets closer to free end, the vibration characteristics of cracked beam seem like intact beam. It is observed the results; the distribution of the material plays an important role on the vibration characteristics.

In figure 6, the effect of crack locations ($L_1/L$) from the left end on the dimensionless fundamental frequency $\bar{\omega}_1$ of edge cracked FGM beams ($a/h = 0.6$) are shown for $E_R = 2$ and $E_R = 0.5$.

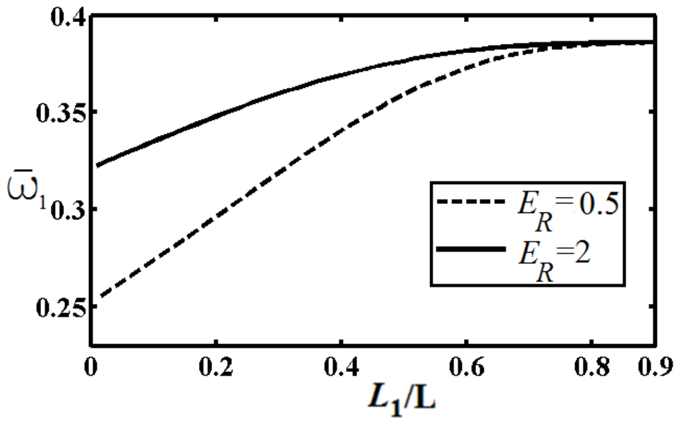


**Figure 6** The effect of crack locations ($L_1/L$) on the dimensionless fundamental frequency $\bar{\omega}_1$ for $E_R = 2$ and $E_R = 0.5$.

It is seen from figure 6 that there is a significant difference between $E_R = 2$ and $E_R = 0.5$ in case of the lower value of $L_1/L$. It is mentioned before in the results of the figure 4 that the dimensionless fundamental frequencies of Young's modulus ratios $E_R = 2$ and $E_R = 0.5$ are same in the case of intact beam situation. In case of the crack situation, the dimensionless fundamental frequencies of Young's modulus ratios $E_R = 2$ and $E_R = 0.5$ are different. It is seen figure 6 that with the crack gets closer to free end, the difference between of $E_R = 2$ and $E_R = 0.5$ decrease. Because, with the crack gets closer to free end, the effect of crack on the vibration characteristics of the beam decrease and the cracked beam seems like intact beam. It is observed the results; the crack locations have a great influence on the vibration characteristics of the FGM beam.

In figure 7, the effect of the crack depth ratio $a/h$ on the dimensionless fundamental frequency $\bar{\omega}_1$ of edge cracked FGM beams ($L_1/L = 0.05$) are shown for $E_R = 2$ and $E_R = 0.5$.

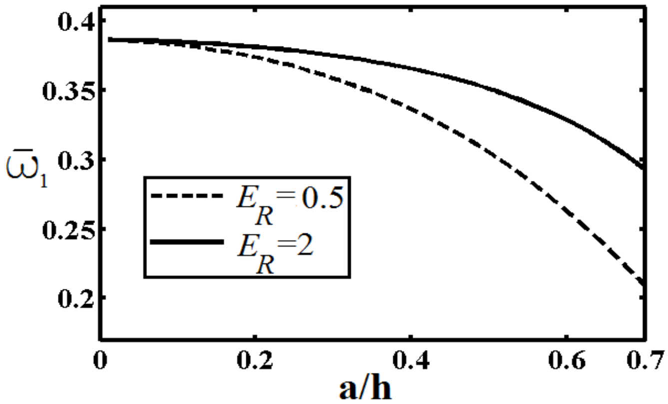


**Figure 7** The effect the crack depth ratio $a/h$ on the dimensionless fundamental frequency $\bar{\omega}_1$ for $E_R = 2$ and $E_R = 0.5$.

It is seen from figure 7 that with increase in the crack depth ratio $a/h$, the difference between the dimensionless fundamental frequency of $E_R = 2$ and $E_R = 0.5$ increases. Although the dimensionless fundamental frequencies of Young's modulus ratios $E_R = 2$ and $E_R = 0.5$ are same in the intact beam, there is a significant difference between with $E_R = 2$ and $E_R = 0.5$ for different crack depth ratio $a/h$.

Figure 8 displays the effect of crack depth ratio $a/h$ on the first and second normalized vibration mode shapes for $E_R = 2$, $L_1/L = 0.05$. Figure 9 displays the effect of the crack location $L_1/L$ on the first and second normalized vibration mode shapes for $E_R = 2$, $a/h = 0.6$.

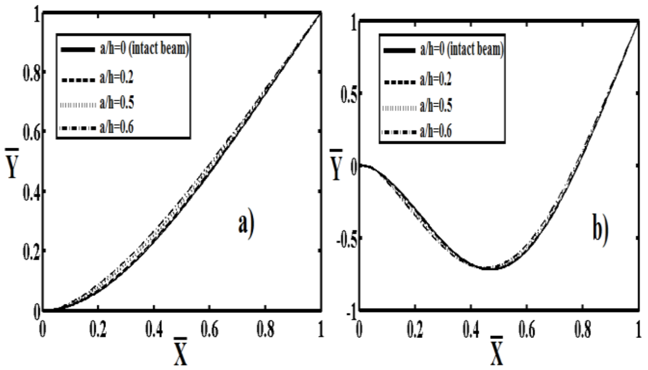


**Figure 8** The effect the crack depth ratio $a/h$ on the a) first and b) second normalized vibration mode shapes.

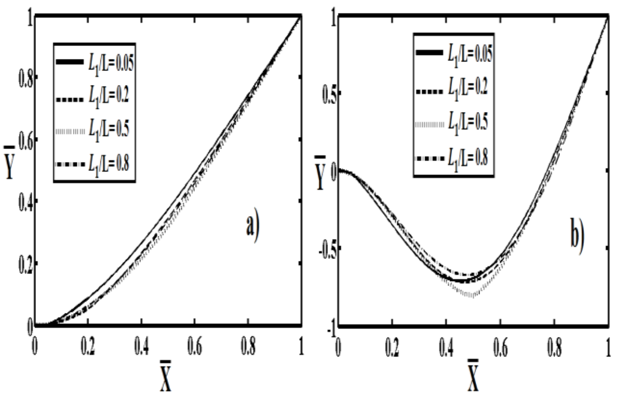


**Figure 9** The effect the crack location $L_1/L$ on the a) first and b) second normalized vibration mode shapes.

It is seen from figure 8 and figure 9 that the crack depth ratio $a/h$ and the crack location $L_1/L$ play important role on the vibration mode shapes.

## IV. CONCLUSIONS

Free vibration analysis of an edge cracked FGM cantilever beam is investigated within the Euler-Bernoulli beam theory by using finite element method. Material properties of the beam change in the thickness direction according to an exponential function. The differential equations of motion are obtained by using Hamilton's principle. The cracked beam is modeled as an assembly of two sub-beams connected through a massless elastic rotational spring. The obtained results are in a very good harmony with the related available results in the literature. The influences of the location of crack, the depth of the crack and different material distributions on the natural frequencies and the mode shapes of the FGM beams are examined in detail.

The following conclusions are reached from the obtained results:
(1) The crack locations and the crack depth have a great influence on the vibration characteristics of the FGM beam.
(2) The distribution of the FGM plays an important role on the vibration characteristics.
(3) There are significant differences of the mechanical behaviour for the cracked and intact FGM beams.
(4) With the suitable choice of the distribution of the FGM, the negative effects of the crack on the beam can be reduced.

**Appendix**

The interpolation functions for axial degrees of freedom are

$$\varphi^{(U)}(X)=[\varphi_1^{(U)}(X)\ \ \varphi_2^{(U)}(X)]^T, \qquad (A1)$$

Where

$$\varphi_1^{(U)}(X)=\left(-\frac{X}{L_e}+1\right), \qquad (A2)$$

$$\varphi_2^{(U)}(X)=\left(\frac{X}{L_e}\right), \qquad (A3)$$

The interpolation functions for transverse degrees of freedom are

$$\varphi^{(V)}(X)=[\varphi_1^{(V)}(X)\ \ \varphi_2^{(V)}(X)\ \ \varphi_3^{(V)}(X)\ \ \varphi_4^{(V)}(X)]^T, \qquad (A4)$$

Where

$$\varphi_1^{(V)}(X)=\left(1-\frac{3X^2}{L_e^2}+\frac{2X^3}{L_e^3}\right), \qquad (A5)$$

$$\varphi_2^{(V)}(X)=\left(-X+\frac{2X^2}{L_e}-\frac{X^3}{L_e^3}\right), \qquad (A6)$$

$$\varphi_3^{(V)}(X)=\left(\frac{3X^2}{L_e^2}-\frac{2X^3}{L_e^3}\right), \qquad (A7)$$

$$\varphi_4^{(V)}(X)=\left(\frac{X^2}{L_e}-\frac{X^3}{L_e^3}\right), \qquad (A8)$$

Where $L_e$ indicates the length of the finite beam element.

The components of the mass matrix $[M]$: $[M_U]$, $[M_V]$, $[M_\theta]$ and $[M_{U\theta}]$ are as follows

$$[M_U]=\int_0^{L_e} I_1[\varphi^{(U)}]^T[\varphi^{(U)}]dX \qquad (A9)$$

$$[M_V]=\int_0^{L_e} I_1[\varphi^{(V)}]^T[\varphi^{(V)}]dX \qquad (A10)$$

$$[M_\theta]=\int_0^{L_e} I_3\left[\frac{d\varphi^{(V)}}{dX}\right]^T\left[\frac{d\varphi^{(V)}}{dX}\right]dX \qquad (A11)$$

$$[M_{U\theta}]=-\int_0^{L_e} I_2\left(\left[\varphi^{(U)}\right]^T\left[\frac{d\varphi^{(V)}}{dX}\right]+\left[\frac{d\varphi^{(V)}}{dX}\right]^T\left[\varphi^{(U)}\right]\right)dX \qquad (A12)$$

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