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Linear Vibration of Rectangular Plate Resting on Translational And Rotational Supports At All Edges

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Abstract *- In this paper, the linear free vibrations analysis of rectangular plate resting on translational and rotational supports at all edges is performed by the semi-analytical method. The Rayleigh-Ritz method is used to investigate linear frequencies and associated mode shapes. The trial plate functions are taken as products of beam function with appropriate end conditions in x and y-direction. These beam functions have the form of the solution of the differential equation, which governs their vibrations. A symbolic calculation was used to find the transcendental equation solving the frequency parameters and mode shapes, and the numerical calculation has allowed finding the numerical beam results. The method used in this work does not respect the plate boundary conditions. However, the results of classical boundary conditions, including the guided case, were in good agreement with the bibliography. Moreover, different finite values of the torsional and translation springs stiffness are studied for several values of the aspect ratio. The results found were compared with the available bibliography; both results match with each other very well.*

Keywords : *Rectangular plates, Elastically Restraints, Rayleigh-Ritz method, Linear vibration, Frequency parameters.*

I. INTRODUCTION

A large number of civil engineering structures are modeled by plates: slab bridges, foundation, floor, lock-gates, and bridge decks. Plates are indispensable in the aerospace and naval sectors: the hull of a ship, its deck and its superstructure, the wings, and a large part of the fuselage of an aircraft. Plates are also present in the machine and mechanical devices [1]. The study of rectangular plate transverse vibrations was begun for over two centuries by Chladni [2]. The motion differential equation of the plate transverse vibrations has an analytical solution only for the plate simply supported or/and guided at opposite edges, and Levy was the first to do this work in 1899 [3]. Rayleigh-Ritz method [4] [5] remains now extensively used in the free vibration analysis of rectangular plates due to their flexibility and conceptual simplicity. These types of methods assume a linear combination of trial functions, which are determined by beam or plate boundary conditions, and by minimizing the system energy functional, the unknown coefficients of the trial functions can be determined. The trial functions including for example, beam characteristic functions [6],[7] , boundary characteristic orthogonal polynomials [8], [9],[10], orthogonal plate functions [11], [12], [13], [14], hierarchical trigonometric functions [15], combination of trigonometric function and lower-order polynomials [16],[17], [18], [19], [20] trigonometric and hyperbolic functions [21], [22],[23],[24], [25]. The other type of solutions, called the strong-form based methods [26], take into account the motion differential equation and the boundary conditions, for example, the superposition method [27], [28], [29] generalized Koialovich's theory based on superposition method [30], Fourier series based on the analytical method [31],[32] and the dynamic stiffness method [33]. The rectangular plates resting on elastically restrained edges have been studied in the literature in the last forty decades, as reviewed by Leissa in his famous monograph [34], which remains a good reference, and in 1973 he studied the 21 classical boundary conditions [6]. P. Laura has made his research on the frequency parameters in transverse vibration of rectangular plates with an elastic edge by RRM exploiting polynomial coordinate function, which only respects the beam boundary condition [35],[36]. This solution was improved by Zhou by putting sine series and third-order polynomial as the set of static beam functions [16],[37], W.L.Li had dealt with these vibrations, by RRM using Fourier series methods, in his solution, the plate boundary conditions are respected [17],[31],[38]. Guo in 2010 had improved the Fourier series method for applying the RRM respecting the plate boundary conditions. Gorman had developed the superposition method [27],[28] witch respected the motion differential equation (1980,2005). The classical dynamic stiffness method was used by X.Liu [26]; it is an improved method of the superposition method. When Li [17] made his research on vibration analysis of rectangular plates with general elastic boundary supports [17], he claimed that " Although beam functions can be generally obtained as a linear combination of trigonometric and hyperbolic functions, they include some unknown parameters that have to be determined from the boundary conditions. Consequently, each boundary condition basically leads to a different set of beam functions. In real applications, this is clearly inconvenient, not to mention the tediousness of determining the characteristic functions for a generally supported beam ". and X.Liu [26] has claimed that the RRM may become numerically unstable. In this paper, vibrations of rectangular plates with elastic four edges were studied with the RRM. The deflection—function was expressed as a product of trigonometric and hyperbolic beam functions in the x and y-direction. This method does not satisfy the boundary conditions of the plate elastically restrained, but it satisfies the end conditions of appropriate beams. Contrary to what Li thought [17], there are no difficulties associated with numerical instability, and the trial functions form a complete set, the accuracy and convergence of the corresponding solution are easily estimated, this is thanks to the symbolic calculation of Matlab. In addition to that, this solution allows easily to deal with rather a difficult problem such as nonlinear vibration of rectangular plate resting on elastic supports at all edges, and it can be readily extended to other more complicated boundary conditions such as concentrated masses, partial supports, point support, orthotropic plates, FGM and sandwich plates. In the next section, the semi-analytical method formulation is exposed to a detailed general approach of the transversal vibrations of the plates with elastically restrained edges in order to find linear frequency parameters and their mode shapes. Linear frequency parameters of the plate were calculated for different plate parameters and compared with available known results. The paper ends with a discussion of the novelty of the solution and its extensions.

II. General formulation

The aim of the present section is to find the linear frequency parameters using the Rayleigh-Ritz method (RRM) for various values of the plate aspect ratio, translational and rotational stiffness of elastic support. The deflection magnitude W is supposed to have the same order as the thickness *H* of the plate, and all strain components are supposed small. The plate trial function used in the RRM is assumed as a product of the trial beam function in x-and y-direction while respecting the plate boundary conditions. First; the trial beam functions are investigated in the fellow's subsection

A. Trial beam functions for beams connected at their ends to translational and rotational stiffness

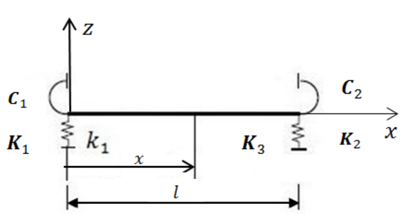


Fig 1: The investigated beam

Consider a beam shown in Fig. (1), and it is connected at its ends to translational and rotational stiffness. The translational stiffness is K_1 and K_2 at the left and the right ends, respectively, the rotational stiffness is C_1 and C_2 at the left and the right ends, respectively. The differential equation which governs the equation of the beam transverse vibration is given by Igora [39]:

$$\frac{d^4Z(x)}{dx^4} + \beta Z(x) = 0 \quad (1)$$

In witch Z is the transverse displacement, x is the abscissa of a current point P, β is the frequency parameters such as $\beta^4 = \omega_b^2 \frac{\mu S}{EI}$, ω_b being the natural beam frequencies, EI is the beam flexural rigidity, S is the cross-section area, and μ is the mass per unit length of the beam. The general solution of Eq. (1) is expressed by [39]:

$$Z_i(x) = c_1\sin(\beta_i x) + c_2\cos(\beta_i x) + c_3\sinh(\beta_i x) + c_4\cosh(\beta_i x) \quad (2)$$

The beam frequency parameters β_i and the integration constants c_1 to c_4 are to be determined by the end conditions. The transverse shearing forces $\mathcal{T}$ and the bending moment $\mathcal{M}$ of the beam are given by [40] :

$$\mathcal{T} = -EI\frac{\partial^3 Z(x)}{\partial x^3} \quad \mathcal{M} = EI\frac{\partial^2 Z(x)}{\partial x^2} \quad (3)$$

The end conditions are expressed at $x = 0$ and $x = l$ by [39], [40] : -

$$\begin{aligned} &\text{At } x = 0 \quad \mathcal{T} = K_1 Z(x) \quad \text{and} \quad \mathcal{M} = -C_1\frac{\partial Z(x)}{\partial x} \\ &\text{At } x = l \quad \mathcal{T} = -K_2 Z(y) \quad \text{and} \quad \mathcal{M}_y = C_2\frac{\partial Z(x)}{\partial x} \end{aligned} \quad (4)$$

l being the beam length. By substituting Eq.(2) into the boundary conditions (4), at $x = 0, l$

$$\begin{aligned} \frac{\partial^3 Z}{\partial x^3} &= \varepsilon\frac{K_1}{EI}Z \\ \frac{\partial^2 Z}{\partial x^2} &= \varepsilon\frac{C_1}{EI}\frac{\partial Z}{\partial x} \end{aligned} \quad (5)$$

$\varepsilon = -1$ when $x = 0$ and $\varepsilon = l$ when $x = 0$. In order to give a dimensionless written to Eel (5), the dimensionless abscissa x^* and the dimensionless deflection Z^* are defined by $x^* = \frac{x}{l}$ and $Z^* = \frac{Z}{H}$, at $x = 0, l$ Eq (5) becomes

$$\begin{aligned} \frac{\partial^3 Z^*}{\partial x^{*3}} &= \varepsilon K^* Z^* \\ \frac{\partial^2 Z^*}{\partial x^{*2}} &= \varepsilon C^*\frac{\partial Z^*}{\partial x^*} \end{aligned} \quad (6)$$

it leads to dimensionless rotational and translational stiffness at both ends:

$$C_{1,2}^* = \frac{C_{1,2}l}{EI} \quad K_{1,2}^* = \frac{K_{1,2}l^3}{EI} \quad (7)$$

As mentioned above, $\varepsilon = -1$ when $x = 0$ and $\varepsilon = l$ when $x = 0$. Eq (6) can be expressed by a matrix writing:

$$[\mathbb{B}]\{c\} = \{0\} \quad (8)$$

where $\{c\}$ is an unknown vector such as $\{c\}^T = \{c_1, c_2, c_3, c_4\}$ and $[\mathbb{B}]$ is a 4×4 square matrix, expressed by:

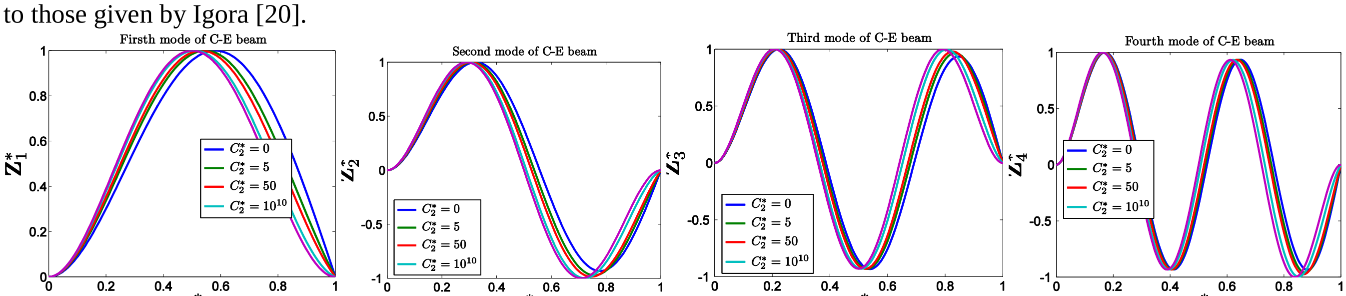
$$[\mathbb{B}] = \begin{bmatrix} -\lambda^3 & K_1^* & \lambda^3 & K_1^* \\ -C_1^* & -\lambda & -C_1^* & \lambda \\ -K_2^*\sin(\lambda) - \lambda^3\cos(\lambda) & -K_2^*\cos(\lambda) + \lambda^3\sin(\lambda) & -K_2^*\sinh(\lambda) + \lambda^3\cosh(\lambda) & -K_2^*\cosh(\lambda) + \lambda^3\sinh(\lambda) \\ C_2^*\cos(\lambda) - \lambda\sin(\lambda) & -C_2^*\sin(\lambda) - \lambda^3\cos(\lambda) & C_2^*\cosh(\lambda) + \lambda\sinh(\lambda) & C_2^*\sinh(\lambda) + \lambda\cosh(\lambda) \end{bmatrix} \quad (9)$$

This matrix contains four dimensionless rotational $C^*_{1,2}$ and translational $K^*_{1,2}$ Parameters at both ends, and also it contains the unknown beam frequency parameter $\lambda = l.\beta$. By Matlab code; a symbolic calculation allowed to easily create the matrix $[\mathbb{B}]$. The non-trivial solutions for vector component $\{c_1, c_2, c_3, c_4\}$ Exist if the determinant of this matrix is equal to zero. A symbolic calculate used to find the determinant expression, and a numerical calculate allowed to solve the roots $(\lambda)_i$ of the obtained frequency equation, corresponding to various modes $\mathbf{Z}_i^*$ and to different values of the dimensionless translational and rotational stiffness, $\mathbf{K}^*_{1,2}$ and $\mathbf{C}^*_{1,2}$ Respectively.

As an illustration, Table (1) lists and compare with the fundamental frequency parameters $\lambda_1^2 = \frac{\omega_b l^2}{\pi}\sqrt{\frac{\mu.S}{EI}}$ of beams clamped (C) at end $x = 0$ and elastically restrained at end $x = l$, the rotational stiffness C_2^* takes three values, 0,100 and 1000, and the translational stiffness K_2^* takes four values 0,100,1000 and 10000. The results found here are identical to those given by Igora [20].

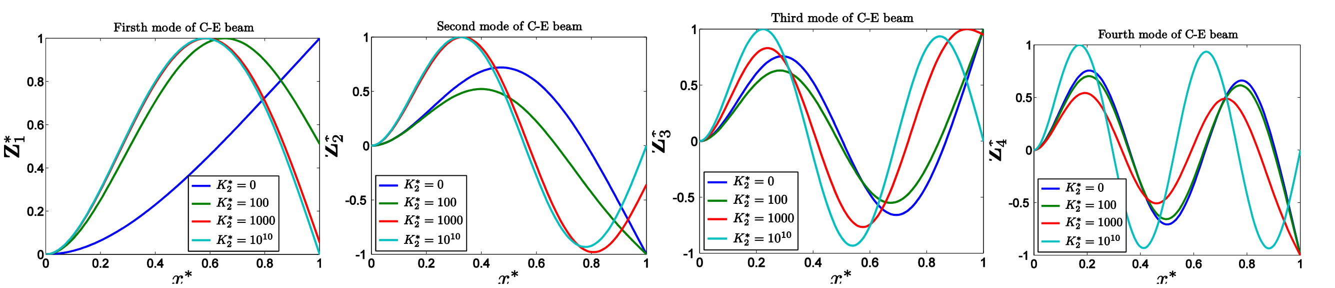
Table 1: Fundamental frequency parameter $\lambda_1^2 = \frac{\omega_b l^2}{\pi}\sqrt{\frac{\rho.S}{EI}}$ For a CE beam with different combinations of the translational and rotational stiffness at x = l.

		$C_2^* = 0$	$C_2^* = 10^2$	$C_2^* = 10^4$
$K_2^* = 0$	Present	1.8751	2.3564	2.3649
	Ref [39]	1.8751	2.3564	2.365
$K_2^* = 10^2$	Present	3.6405	3.8403	3.8482
	Ref [39]	3.6454	3.8403	3.8482
$K_2^* = 10^3$	Present	3.8978	4.5845	4.6243
	Ref [39]	3.8978	4.5845	4.6243
$K_2^* = 10^4$	Present	3.9237	4.6754	4.7193
	Ref [39]	3.9237	4.6754	4.7193



First mode **Second mode** **Third mode** **Fourth mode**

Fig 2: First four normalized modes of CE beam, with $K^* = \infty$ and with several values of rotational stiffness C_2^*



First mode **Second mode** **Third mode** **Fourth mode**

Fig 3: First four normalized modes of CE beam, with $K^* = \infty$ and with several values of rotational stiffness C_2^*

As illustration also, Figs. (2-3) show the rotational and translational stiffness influence on the first four normalized modes of CE beam. For Fig.(2), the translational stiffness K_2^* tends toward infinity, and the rotational stiffness C_2^* takes four values $= 0,5,50,10^{10}$. For Fig.(3), the rotational stiffness C_2^* tends toward infinity, and the translational stiffness K_2^* takes four values $= 0,100,1000,10^{10}$.

B. General formulation

The thin rectangular plate shown in Fig (4) is resting on translational and rotational supports at its all edges, for clarity, translational spring is the only one drawn at the edge $x = a$, and rotational spring is drawn at $x = b$. It has a length a, and a width b.

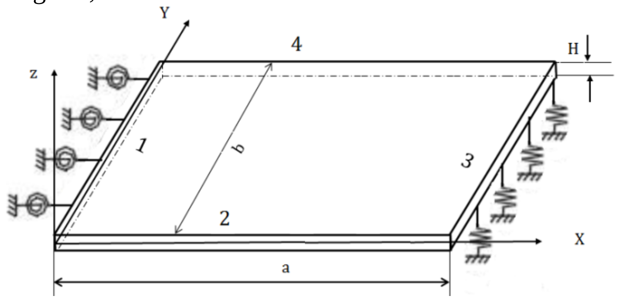


Fig 4: The investigated rectangular plate resting on translational and rotational supports at all edges.

The translational stiffnesses per unit length of the edges $x = 0$, $y = 0$, $x = a$ and $y = b$, are K_1, K_2, K_3, K_4 respectively, and the rotational stiffnesses at the edges $x = 0$, $y = 0$, $x = a$, and $y = b$, are C_1, C_2, C_3, C_4 respectively. $\boldsymbol{W}(x, y, t)$ denotes the transverse displacement of the current point $M(x, y)$ of the middle plane. Under the assumption of neglected the rotary inertia, the plate kinetic energy T is expressed as in [21], [44]:

$$T = \frac{1}{2}\rho H \int_S \left(\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{t}}\right)^2 dS \tag{10}$$

In which H is the plate thickness and ρ is the mass density per unit area of the beam. The plate total strain energy V_T is the sum of the strain energy due to the bending V_bAnd the strain energy stored by the elastic edge restraints V_{Edg} : $(V_T = V_b + V_{Edg})$ are given by [7], [21], [23], [41].

$$V_b = \frac{D}{2}\int_S \left(\frac{\partial^2 \boldsymbol{W}}{\partial x^2} + \frac{\partial^2 \boldsymbol{W}}{\partial y^2}\right)^2 + 2(1-\nu)\left(\left(\frac{\partial^2 \boldsymbol{W}}{\partial x \partial y}\right)^2 - \frac{\partial^2 \boldsymbol{W}}{\partial x^2}\frac{\partial^2 \boldsymbol{W}}{\partial y^2}\right) dS \tag{11}$$

in which $D = \frac{EH^3}{12(1-\nu^2)}$ Is the plate flexural rigidity, E being Young's modulus and ν being Poisson's ratio (in all follows $\nu = 0.3$). The strain energy stored by the elastic edge restraints V_{Edg} is the sum of the translational and rotational elastic edge restraints V_{Edg}^{Tra} and V_{Edg}^{Rot} respectively [34].

• The strain energy V_{Edg}^{Tra} stored by the translational elastic edge restraints at all edges is:

$$V_{Edg}^{Tra} = \frac{1}{2}\int_0^b \left(K_1.(\boldsymbol{W}^2)_{x=0} + K_3(\boldsymbol{W}^2)_{x=a}\right)dy + \frac{1}{2}\int_0^a \left(K_2(\boldsymbol{W}^2)_{y=0} + K_4(\boldsymbol{W}^2)_{y=b}\right)dx \tag{12}$$

• The strain energy V_{Edg}^{Rot} stored by the rotational elastic edge restraints at all edges is:

$$V_{Edg}^{Rot} = \frac{1}{2}\int_0^b \left(C_1\left(\frac{\partial \mathbf{W}}{\partial x}\right)^2_{x=0} + C_3\left(\frac{\partial \mathbf{W}}{\partial x}\right)^2_{x=a}\right) dy + \frac{1}{2}\int_0^a \left(C_2\left(\frac{\partial \boldsymbol{W}}{\partial y}\right)^2_{y=0} + C_4\left(\frac{\partial \boldsymbol{W}}{\partial y}\right)^2_{y=b}\right) dx \tag{13}$$

The plate displacement functions $\mathbf{W}(x, y, t)$; calculated under the assumptions of harmonic motion and the spectral expansion is expressed by [21] :

$$\mathbf{W}(x, y, t) = a_k w_k \sin(\omega . t) \tag{14}$$

where

$$w_k(x, y) = X_i(x)Y_j(y) \quad k = n(i-1) + j \tag{15}$$

in which t is the time, ω is the angulaire frequency, $X_i(x)$ and $Y_j(y)$ are the trial beam functions that have the same boundary conditions in the x-and y-direction, respectively. If the number of functions $X_i(x)$ and $Y_j(y)$ are m and n respectively, the number of the plate functions is $N = n \times m$, $i = 1 \ldots m$, $j = 1 \ldots n$. X_i and Y_j have to verify the beam end conditions cited above. a_k being the contribution coefficient of the function spatial w_k. The discretized expressions of the kinetic and strain energy expressed in Eqs. ((10)-(13) are:

$$T = \frac{1}{2}\omega^2 . a_i . a_j . m_{ij} \cos^2(\omega t)$$
$$V_b = \frac{1}{2} a_i a_j k_{ij}^b \sin^2(\omega t) \tag{16}$$
$$V_{Edg} = \frac{1}{2} . a_i . a_j . k_{ij}^{Edg} . \sin^2(\omega t)$$

In which m_{ij} and k_{ij}^b are the mass and bending tensors, respectively. These tensors are given by [21] [42]:

$$m_{ij} = \rho H \int_S w_i w_j dS \tag{17}$$

$$k_{ij}^b = D\int_S \left(\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2}\right)\left(\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2}\right) + 2(1-\nu).\left(\frac{\partial^2 w_i}{\partial x \partial y}.\frac{\partial^2 w_j}{\partial x \partial y} - \frac{\partial^2 w_i}{\partial x^2}.\frac{\partial^2 wj}{\partial y^2}\right) dS \tag{18}$$

and k_{ij}^{Edg} represent the rigidity tensor associated with the energy stored in the elastic edge restraints, and it is the sum of the rigidity tensor due to the translational k_{ij}^{Tra} and rotational k_{ij}^{Rot} elastic edge restraints: $k_{ij}^{Edg} = k_{ij}^{Tra} + k_{ij}^{Rot}$:

$$\begin{aligned} k_{ij}^{Tra} = & \int_0^b \left((K_1.w_i w_j)_{x=0} + (K_3 w_i w_j)_{x=a}\right) dy \\ + & \int_0^a \left((K_2.w_i w_j)_{y=0} + (K_4 w_i w_j)_{y=b}\right) dx \\ k_{ij}^{Rot} = & \int_0^b \left(\left(C_1 \frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}\right)_{x=0} + \left(C_3 \frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}\right)_{x=a}\right) dy \\ + & \int_0^a \left(\left(C_2 \frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}\right)_{y=0} + \left(C_4 \frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}\right)_{y=b}\right) dx \end{aligned} \tag{1}$$

One put k_{ij} the total rigidity tensor:

$$k_{ij} = k_{ij}^b + k_{ij}^{Edg} \tag{20}$$

Hamilton's principle governs the linear plate vibration [21]:

$$\delta \int_0^{\frac{2\pi}{\omega}} (V_T - T)dt = 0 \tag{21}$$

After calculations the one gets a linear eigen value problem, written in a matrix form as [21], [23].

$$2[K]\{A\} - 2\omega^2[M]\{A\} = \{0\} \qquad (22)$$

[K] and [M] are the matrices associated with the tensors defined above? {A} is the column vector of the basic function contribution coefficients. These parameters have been calculated numerically by a cod Matlab program. Eq. (22) is the classical eigenvalue problem corresponding to the RRM of linear vibrations. If one defines, as in Ref [41], the non-dimensional parameters:

$x^* = \frac{x}{a}, y^* = \frac{y}{b}$: dimensionless abscissa and ordinate, respectively.

$w^* = \frac{w}{H}$: dimensionless deflection.

$K_i^* = K_i \frac{a^3}{D}$: dimensionless translational stiffness at edge $x = 0, a$ $(i = 1,3)$

$K_i^* = K_i \frac{b^3}{D}$: dimensionless translational stiffness at edge $y = 0, b$ $(i = 2,4)$

$C_i^* = C_i \frac{a}{D}$: dimensionless rotational stiffness at edge $x = 0, a$ $(i = 1,3)$

$C_i^* = C_i \frac{b}{D}$: dimensionless rotational stiffness at edge $y = 0, b$ $(i = 2,4)$

it leads to the frequency parameter Ω to be determined in the next subsection:

$$\Omega = a^2 \sqrt{\frac{\rho H}{D}}\,\omega \qquad (23)$$

Eq.(24) becomes :

$$2[K^*]\{A\} - 2\Omega^2[M^*]\{A\} = \{0\} \qquad (24)$$

The general terms of the dimensionless mass $[M^*]$ and the linear rigidity $[K^*]$ matrices are expressed by:

$$m_{ij}^* = \int_S w_i^* . w_j^* dxdy$$

$$k_{ij}^* = \int_S \frac{\partial^2 w_i^*}{\partial x^{*2}} \frac{\partial^2 w_j^*}{\partial x^{*2}} + \alpha^2 \left(\frac{\partial^2 w_i^*}{\partial x^{*2}} \frac{\partial^2 w_j^*}{\partial y^{*2}} + \frac{\partial^2 w_i^*}{\partial y^{*2}} \frac{\partial^2 w_j^*}{\partial x^{*2}} \right) + \alpha^4 \frac{\partial^2 w_i^*}{\partial y^{*2}} \frac{\partial^2 w_j^*}{\partial y^{*2}} + 2(1-\mu)\alpha^2 \left(\frac{\partial^2 w_i^*}{\partial x^* \partial y^*} \frac{\partial^2 w_j^*}{\partial x^* \partial y^*} - \frac{\partial^2 w_i^*}{\partial x^{*2}} \frac{\partial^2 w_j^*}{\partial y^{*2}} \right) dxdy$$

(25)

where $\alpha = \frac{a}{b}$ is the plate aspect ratio. Eq.(24) represents the dimensionless Rayleigh-Ritz formulation of the linear vibration problem. Its solution yields the i^{th} frequency parameters are given by Eq.(23) together with the i^{th} mode shapes $w_i(x, y)$ expressed by Eq (15).

III. NUMERICAL RESULTS AND DISCUSSIONS

Eq. (24) allows finding the frequency parameters and associated mode shapes using the Matlab program for several examples dealing with various aspect ratio $\alpha = \frac{a}{b}$ and different value of the eight dimensionless translational and rotational stiffness K^* and C^*, (i = 1 to 4). In order to have a concise presentation, only selective and representative results are presented. The classical edge boundary conditions are obtained by giving appropriate values to parameters K^* and C^*, for clamped edge (C) K^* and C^* tend toward infinity, for simply supported edge (S) K^* tends toward infinity and $C^* = 0$, for guided edge (G) $K^* = 0$ and C^* Tends toward infinity, and for free edge, these stiffnesses are void. The letter (E) designates edges with finite and non-zero stiffness. Four capital letters designate the plate boundary conditions; the first and the third letters designate the edge types at $x = 0$ and at $x = a$, respectively, and the second and the fourth letters designate the edge types at $y = 0$ and at $y = b$, respectively.

A. Convergence study

To examine the solution convergence of the RRM described by Eq. (24) for a rectangular CCCC plate, in particular, the three lowest frequency parameter Ω_i, i = 1 to 3. The relative difference is defined by:

$$\Delta_i\% = \frac{\Omega_i - \Omega_{ref_i}}{\Omega_{ref_i}} \times 100 \qquad (26)$$

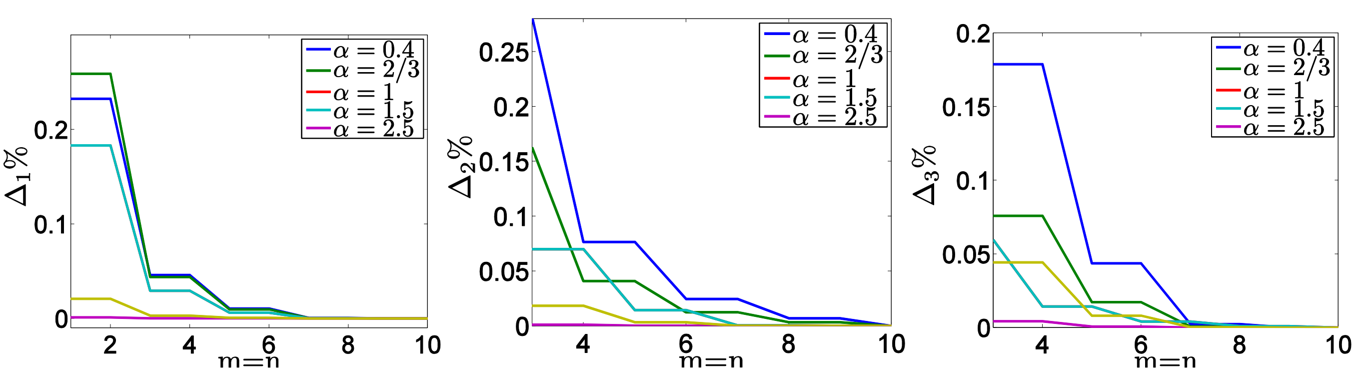


Fig 5: The three relative differences Δ_i versus numbers of the trial beam functions m = n with several aspect ratio α for CCCC plates.

In which Ω_{Ref_i}, i = 1 to 3, are the first four frequency parameters given as references (Leissa [6]). Fig (5) gives the curves of the relative difference Δ_i versus the number m=n with several values of aspect ratio α=a/b, m and n are the numbers of beam functions used in RRM in x-and y-direction, respectively. For the first mode, one trial function in each direction can be safely used to obtain accurate results; for the second and third modes, 3 trial beam functions give excellent results with a relative difference of less than 0.3%. Fortunately, the results of the convergence study of the other boundary conditions are fastest than the CCCC case, and the value $m = n = 5$ is correct for all boundary conditions and modes carried out in this work.

B. Frequency parameters

Table (2) shows five sets of results for square plates with different boundary conditions CCCC, CCSC, CCCG, CSGF, and CGCG. The results are presented by the first eight frequency parameters $\Omega_k = \omega_k . a^2\sqrt{\rho . H/D}$, They are compared with those given by X.Liu Ref [26] and obtained by spectral-dynamic stiffness method (S-DSM) and finite element method (FEM).

Table 2: First eight frequency parameters $\Omega_k = \omega_k . a^2\sqrt{\rho . H/D}$ of square plates with several boundary conditions

		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8
CCCC	Present	35.9874	73.4002	73.4002	108.228	131.600	132.220	165.044	165.044
	Ref [26] S-DSM	35.9852	73.3938	73.3938	108.217	131.581	132.205	165	165
	Ref [26] FEM	35.985	73.4	73.4	108.22	131.59	132.22	165.01	165.01
CCSC	Present	28.9512	54.7470	69.3274	94.5890	102.229	129.096	140.218	154.794
	Ref [26] S-DSM	28.9509	54.7431	69.327	94.5853	102.216	129.096	140.205	154.776
	Ref [26] FEM	28.951	54.744	69.33	94.588	102.22	129.11	140.21	154.79
	Exact	28.9509	54.7431	69.327	94.5853	102.216	129.096	140.205	154.776
CSGF	Present	6.619	19.958	31.782	47.087	53.633	76.262	80.104	92.871
	Ref [43] RRM	6.601	19.954	31.677	47.034	53.632	76.003		
CCGG	Present	8.996	32.895	33.051	55.010	77.226	77.292	98.197	98.480
	Ref [43] RRM	8.996	32.895	33.051	55.008	77.226	77.291		

The frequency parameters calculated for a square CCSC and SCSC plates were made with $m = n = 10$, (for more accurate) the comparison with the results given by X.Liu [26] is very good. The results for CCCG, CSGF, and CGCG are compared with those given by Monterrbio [43]; the calculated frequencies show an excellent agreement with those given in Ref [26]. In order to study the finite stiffness values, several examples were treated. The first example deals with a square SSSS plate with a uniform elastic restraint against rotation

Table 3: Six first frequency parameters, Ω_k of a square SSSS plate elastically restrained against rotation at all edges, with various values of $C_j^* = C^*$, $j = 1, 2, 3, 4$.

C^*	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1	21.50	51.19	51.19	80.82	100.58	100.59
	21.50 [a]	51.18	51.18	80.82	100.58	100.58
10	28.50	60.22	60.22	90.82	111.19	111.41
	28.49 [a]	60.20	60.20	90.79	111.16	111.39
100	34.59	70.64	70.64	104.25	126.82	127.40
	34.67 [a]	70.77	70.77	104.44	127.01	127.59
1000	35.82	73.06	73.06	107.70	130.98	131.60
	35.84 [a]	73.10	73.10	107.78	131.06	131.68

[a] Results are given by Li Ref [31]

along all edges, which tack various values of the rotational stiffness $C^* = 1,10,100,1000$. The lowest six frequency parameters $\Omega_i = \omega_i . a^2\sqrt{\frac{\rho . H}{D}}$, $i = 1$ to 6 are summarized in Table (3) and compared with those found via an analytical method developed by W.L.Li [31], both results match very well with each other. The weak difference percentage between the two result sets indicates the precision of the RRM carried out here. The second example deals with the influence of the translational and rotational stiffness on the frequency parameters, and the study plate is a square CSES plate with various combinations of the rotational and translational stiffness at $x = a$, $C_3^* = 0,10,100,\infty$ and $K_3^* = 0,10,100,\infty$.

Table 4: Fundamental frequency parameters, Ω_1 for a CSES square plate with various combinations of the rotational and translational stiffness at $x = a$.

C_3^*	$K_3^* = 0$	10	100	∞
0	12.7152	13.9533	19.1946	23.6204
	12.6874 [a]	13.9315	19.2195	23.6463
10	13.4182	14.3526	19.3950	26.5219
	13.4098 [a]	14.346	19.3982	26.5556
100	13.6407	14.4823	19.4745	28.5358
	13.6491 [a]	14.4891	19.4782	28.5523
∞	13.6407	14.4823	19.4745	28.5358
	13.6491 [a]	14.4891	19.4782	28.5523

[a] Results are given by Li Ref [38]

Given by Li [38] using a Fourier series method for plates having two opposite edges simply supported, the difference percentage doesn't exceed 0.22%.

Table 5: Fundamental frequency parameters, Ω_1 of rectangular plate CCCE with $K_4^* = 10^{10}$ and for various values of C_4^*With different aspect ratios.

C_4^*	$\bar{\alpha} = 0.4$	0.66	1	1.5	2.5
			CCCC		
0	17.13	21.41	31.83	58.18	146.50
	17.18	21.44	31.87	58.34	147.08
0.001	17.13	21.41	31.83	58.18	146.50
	17.19	21.44	31.87	58.35	147.08
1	17.95	22.05	32.25	58.40	146.57
	18.00	22.08	32.28	58.53	147.11
3.2	19.19	23.06	32.93	58.77	146.72
	19.24	23.09	32.95	58.88	147.21
10	20.99	24.59	34.04	59.43	147.01
	21.05	24.62	34.05	59.51	147.46
32	22.50	25.94	35.10	60.12	147.37
	22.59	25.99	35.13	60.24	147.93
100	23.23	26.62	35.65	60.51	147.61
	23.31	26.66	35.67	60.61	148.13
∞	23.65	27.01	35.99	60.76	147.78
	23.73	27.04	36.00	60.85	148.28
			SESS		
0	11.45	14.26	19.74	32.08	71.55
	11.46 [a]	14.26	19.75	32.09	71.60
0.001	11.53	14.32	19.79	32.11	71.57
	11.54 [a]	14.33	19.80	32.12	71.61
1	12.19	14.86	20.18	32.36	71.68
	12.20 [a]	14.87	20.19	32.37	71.73
3.2	13.25	15.76	20.88	32.82	71.91
	13.26 [a]	15.77	20.89	32.86	72.01
10	14.69	17.05	21.95	33.60	72.35
	14.70 [a]	17.06	21.97	33.68	72.58
32	15.81	18.11	22.89	34.37	72.88
	15.83 [a]	18.12	22.93	34.49	73.23
100	16.34	18.61	23.37	34.79	73.22
	16.36 [a]	18.63	23.40	34.92	73.61
1000	16.60	18.87	23.62	35.02	73.42
	16.62 [a]	18.88	23.65	35.15	73.81
∞	16.63	18.90	23.65	35.05	73.44
	16.65 [a]	18.91	23.68	35.18	73.84

[a] Results are given by Li Ref [35]

The frequency parameters are not very sensible to the rotational stiffness: for $K_3^* = 100$ it varies from $\Omega = 19.19$ for $C_3^* = 0$ to $\Omega = 19.49$ for $C_3^* = \infty$ but they are influenced by the translational stiffness: for $C_3^* = 100$ it varies from $\Omega = 13.6$ for $K_3^* = 0$to $\Omega = 28.5$ for $K_3^* = \infty$. The following example deals with the influence of the aspect ratio $\bar{\alpha} = \frac{b}{a}$ on the frequency parameters. Rectangular CCCE and SESS plates are investigated. The first plate is clamped at the edges $x = 0, a$, $y = 0$, while the edge $y = b$ is elastically restrained against rotation with several stiffness levels $C_4^* = 0, 0.001, 1, 3.2, 10, 32, 100, \infty$, and it has zero deflection i.e. the translational stiffness K_4^* is token equal to a very big value $1e^{10}$. The second plate is simply supported at all edges in more the edge $y = 0$ is elastically restrained against rotation with several stiffness levels $C_4^* = 0, 0.1, 1, 3.2, 10, 32, 100, 10^4, \infty$.The corresponding fundamental frequency parameters $\bar{\Omega}_1 = \omega_1 . b^2 \sqrt{\frac{\rho . H}{D}}$ Are summarized in Table (5), and compared with those given by Ref [35], the differences remain less than 0.62%. As may be expected, the frequency parameter is very influenced by the aspect ratio. The last example gives the five lowest frequency parameters for a CCCS plate for different value of the aspect ratio $\bar{\alpha} = 0.4, 1, 2.5$, moreover the edge $y = b$ is retained in rotation. the rotational stiffness tacks several values $C_4^* = 0.001, 1, 100$. Table (6) lists and compares the present results with those given by Ref [39]. The comparison is very good just for the first four frequency parameters, but for the fifth mode, this difference becomes acceptable; this can be justified by the fact that the results in tables (5) and (7) of this reference do not respect the bounds given in the same table of their two last columns.

Table 6: Five lowest frequency parameters of rectangular plate CCCE with and $K_4^* = 10^{10}$, $C_4^* = 0.01, 1, 100$ with different aspect ratios.

	$C_4^* = 0.01$		$C_4^* = 1$		$C_4^* = 100$	
$\bar{\alpha}$	Present	Ref[39]	Present	Ref[39]	Present	Ref[39]
1	31.83	31.51	32.23	31.94	35.63	35.43
		(31.87)		(32.28)		(35.67)
	63.35	63.10	64.08	63.83	72.35	71.65
	71.08	69.33	71.26	69.52	73.16	72.16
	100.83	99.42	101.29	99.87	107.32	106.08
	116.40	116.71	117.25	116.94	129.80	126.57
2.5	107.08	106.91	112.19	112.05	145.20	145.25
		(107.41)		(112.52)		(146.24)
	139.67	138.50	143.48	142.42	171.40	170.98
	194.42	190.11	197.05	192.95	219.29	216.53
	270.48	259.49	272.30	261.58	289.86	281.25
	322.57	322.66	328.18	328.27	382.86	362.57
	353.44	343.99	358.56	345.61	387.36	389.69
0.4	23.44	22.25	23.45	22.27	23.62	22.45
		(23.53)		(23.55)		(23.70)
	27.02	25.85	27.06	25.89	27.70	26.57
	33.80	32.69	33.87	32.77	35.18	34.14
	44.13	43.12	44.23	43.22	46.22	45.27
	58.03	56.86	58.15	56.86	60.82	56.99
	62.97	60.96	62.98	60.98	63.07	61.39

[a] Results are given by Li Ref [44], values within parenthese are from Ref [35]

C. Mode shapes

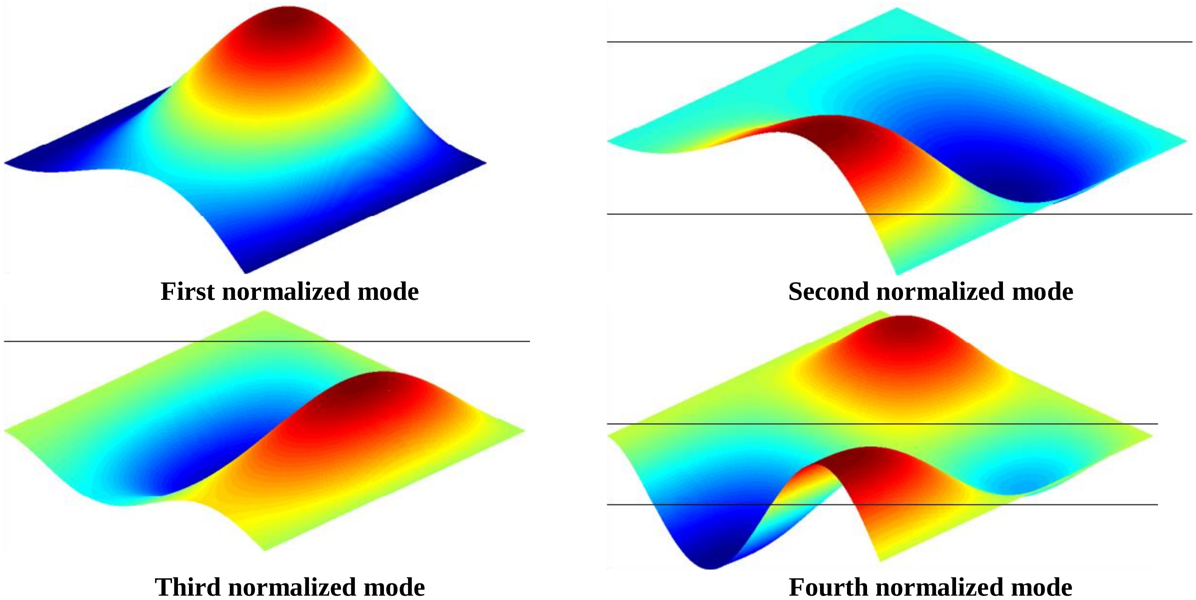
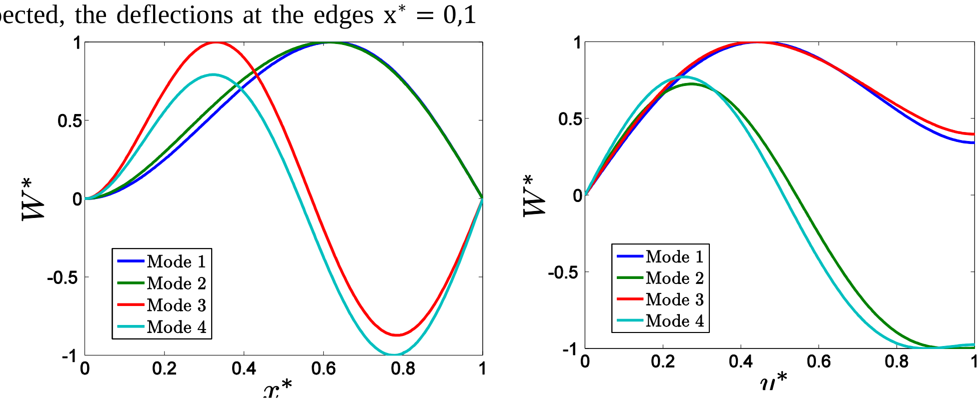


Fig 6: First four mode shapes of a rectangular CSSE with $C_4^* = \infty$, $K_4^* = 200$ and $\alpha = \frac{2}{3}$.

This subsection studies the influence of the aspect ratio, the translational, and the rotational stiffness on the four lowest mode shapes. Fig (6) shows the first four mode shapes of a rectangular CSSE with a rotational stiffness tending to infinity and a translational stiffness, $K_4^* = 200$ at the edge $y = b$. The aspect ratio is $\alpha = \frac{a}{b} = \frac{2}{3}$. The normalized cross-sections of these four modes are plotted in Fig (7): the normalized cross-sections in x-direction $y = \frac{b}{2}$ are plotted in Fig (7.a), and the normalized cross-sections in y-direction $x = \frac{a}{2}$ are plotted in Fig (7.b).

As might be expected, the deflections at the edges $x^* = 0,1$ shown in Fig (7.a) and at edge $y^* = 0$ shown in Fig (7.b) are zero because of the boundary condition natures, i.e., clamped at $x^* = 0$ and simply supported at $x^* = y^* = 1$. Moreover, the slopes at edges $x^* = y^* = 1$ are null since the edge $x = 0$ is clamped and the edge $y^* = 1$ is totally retained in rotation ($C_4^* =$ tends toward infinity). The influence of the aspect ratio on the mode shapes is illustrated by Fig (8); this figure plots the normalized cross-sections of four modes corresponding to the middle plate in x-direction $y = \frac{b}{2}$ in (8.a)



(a): middle plate in x-direction y=b/2, (b) middle plate in y-direction x=a/2

Fig 7: Normalized cross-sections of the four modes corresponding to a rectangular CSSE plate with $C_4^* = \infty$, $K_4^* = 200$ $\alpha = \frac{2}{3}$.

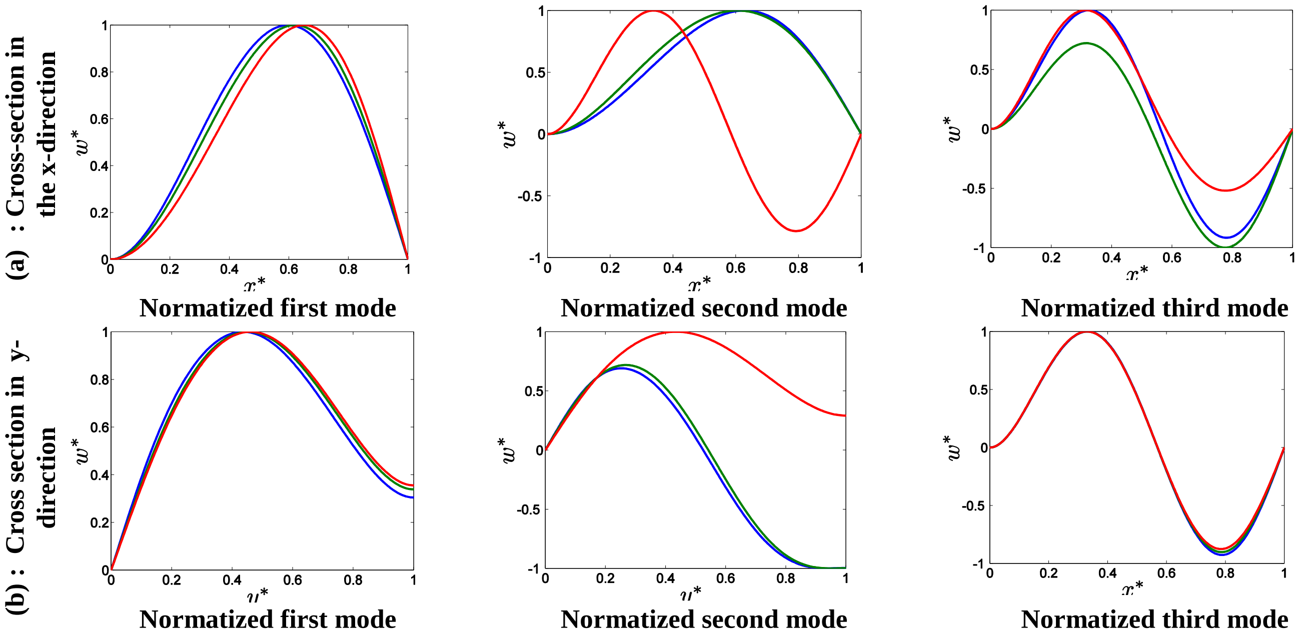


Fig 8: Normalized cross-sections of three modes corresponding to, (a): middle plate in x-direction $y = \frac{b}{2}$, (b) middle plate in y-direction $x = \frac{a}{2}$ for a rectangular CSSE plate with $C_4^* = \infty$, $K_4^* = 200$ and three value of aspect ratio $\alpha = \frac{a}{b}$. Blue curve $\alpha = 0.4$, green curve $\alpha = \frac{2}{3}$, red curve $\alpha = 1$.

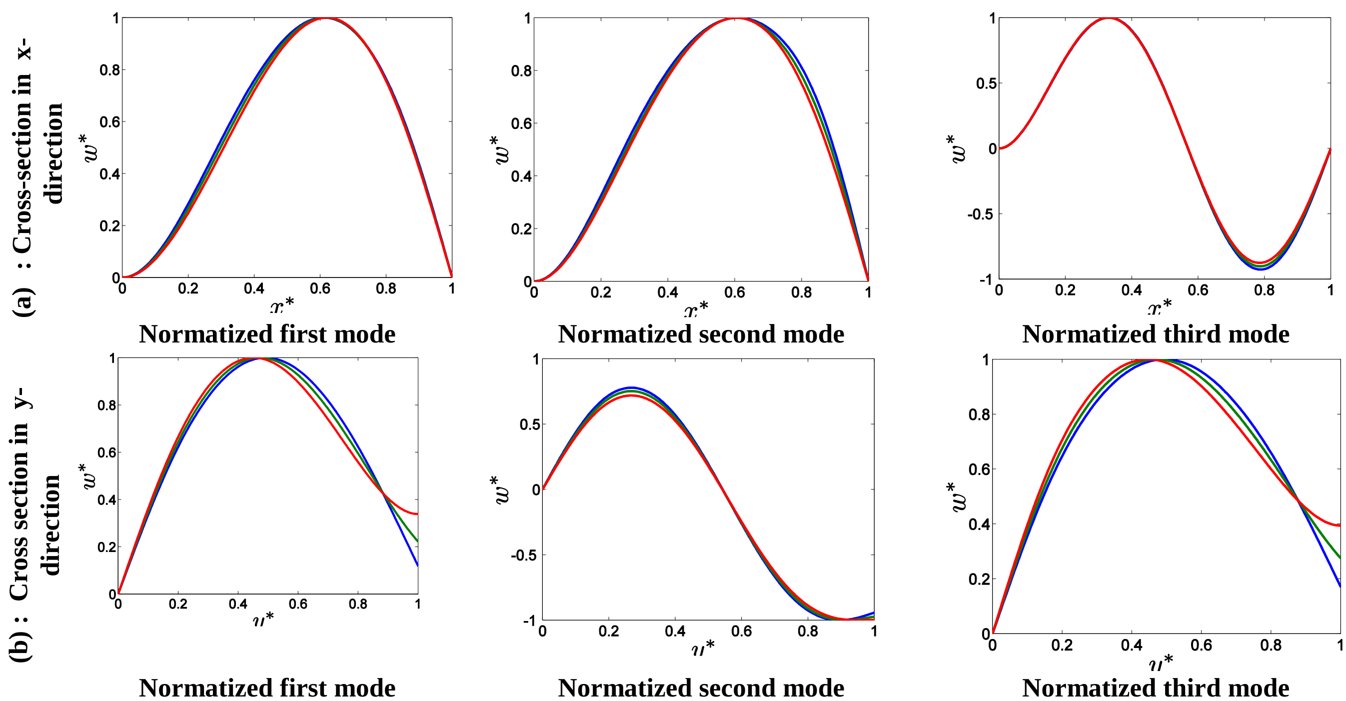


Figure 9: Normalized cross-sections of three modes corresponding to, (a): middle plate in x-direction $y = \frac{b}{2}$, (b) middle plate in y-direction $x = \frac{a}{2}$ for a rectangular CSSE plate with, K_4^*=200 and three value of rotational stiffness C_4^*, the aspect ratio is α=2/3. Blue curve $C_4^* = 0$, green curve $C_4^* = 5$, red curve $C_4^* = \infty$.

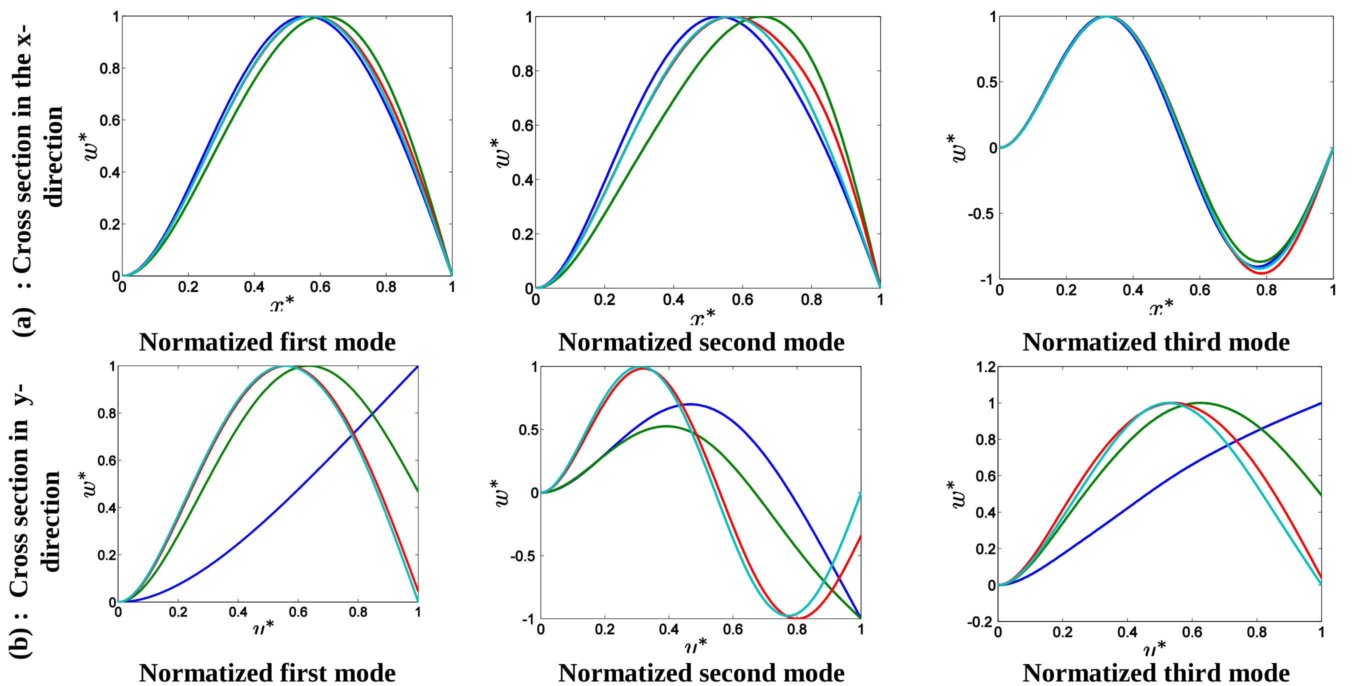


Figure 10: Normalized cross-sections of three modes corresponding to, (a): middle plate in x-direction $y = \frac{b}{2}$, (b) middle plate in y-direction $x = \frac{a}{2}$ for a rectangular CCSE plate with, $C_4^* = 0$ and four values of translational stiffness K_4^*, the aspect ratio is $\alpha = \frac{2}{3}$. Marine blue curve $K_4^* = 0$, green curve $K_4^* = 2000$, red curve $K_4^* = 3000$, sky blue curve $K_4^* = \infty$.

and to the middle plate in y-direction $x = \frac{a}{2}$ in (8.b) for a rectangular CSSE plate with a rotational stiffness $C_4^* = \infty$, a translational stiffness $K_4^* = 200$ at the edge $y = b$ and three value of aspect ratio $\alpha = \frac{a}{b}$. Blue curve $\alpha = 0.4$, green curve $\alpha = \frac{2}{3}$, red curve $\alpha = 1$. The first mode is not very sensitive to the aspect ratio, however from the second one, all modes are influenced by the aspect ratio in x- and y-direction.

The normalized cross-sections of four modes corresponding to the middle plate in x-direction $y = \frac{b}{2}$ and to the middle plate in y-direction $x = \frac{a}{2}$ for a rectangular CCSE plate are plotted in Fig (10). The rotational stiffness is token $C_4^* = 0$ and the translational stiffness K_4^* takes four values 0,100,1000 and a very big value (10^{10}), the aspect ratio is $\alpha = \frac{2}{3}$. The translational stiffness acts a lot on the modes.

IV. CONCLUSION

A Rayleigh-Ritz method is proposed for the transversal vibration analysis of a rectangular plate with elastically restrained edges. The defection trial functions are assumed as products of beam functions witch are the exact solution of differential motion equation of beam with appropriate end conditions, i.e., trigonometric and hyperbolic terms, and the unknown coefficients of these terms are determined from the end conditions. The results show that the influence of translational stiffness on the frequency parameters is greater than that of the rotational stiffness. Even if the trial functions do not verify the boundary condition of the plate, the current method is applicable to the various boundary conditions, including almost all classical cases, thanks to the good accuracy and reliability of its results, which have been proven by several comparisons with the bibliography. It is pointed out that the current method errs considerably if the plates have two opposite edges having small stiffness values of the translational stiffness, in particular two opposite free edges, this remark agrees with the theses. This remark is in agreement with that what has been written by Zhou in his conclusion of the Ref [42]. However, the current method is simple, straightforward, allows to obtain numerical solutions of a rather difficult problem, and it can be readily extended to other more complicated boundary conditions such as non-uniform elastic restraints, concentrated masses, partial supports, orthotropic plates, FGM, and sandwich plates.

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