Some Novel Similarity Measures of Hesitant Fuzzy Sets And Their Applications To MADM

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Abstract — *Similarity measure is a very important topic in fuzzy set theory. Torra proposed the notion of hesitant fuzzy set (HFS) which is a generalization of the notion of Zadeh' fuzzy set. In this paper, a hesitant fuzzy quasi subset is first defined. Moreover, the modified axiom definitions of distance and similarity measures for HFSs are given and some novel distance and similarity measures for HFSs are developed. Based on the proposed similarity measures, a method of multiple attribute decision making (MADM) under hesitant fuzzy environment is established. Additionally, a numerical example is given to illustrate the application of the proposed similarity measures of HFSs in multiple attribute decision making.*

Keywords — *Hesitant fuzzy set, Distance measure, Similarity measure, Multiple attribute decision making.*

I. INTRODUCTION

Since the concept of fuzzy set was introduced by Zadeh [1], many novel methods and theories dealing with imprecision and uncertainty have been studied, such as intuitionistic fuzzy set (IFS) [2], interval-valued fuzzy set (IVFS) [4], vague set [3], and type-2 fuzzy set (T2FS) [5]. They are all extensions of Zadeh' fuzzy set theory. And many scholars have also discussed a lot of meaningful applications of these theories in cluster analysis, multi-criteria decision, grey relational analysis and information aggregation.

Similarity measure is a very important tool and it has been applied in many fields. Since Zadeh [6] introduced the concept of similarity relation, lots of researchers have studies the similarity measures of fuzzy sets from different aspects. Fan and Xie [7] as well as Liu[8] gave an axiom definition of similarity measures of two fuzzy sets and studied some basic properties. Based on intersection and union operations, the maximum difference and the differences as well as the sum of membership grades, Pappis and Karacapilidis [9] investigated three similarity measures of fuzzy sets. In [10], Wang introduced two new similarity measures of fuzzy sets. Moreover, many similarity measures for vague sets, IFSs, IVFSs and T2FSs have also been widely developed [3, 11-23].

Recently, Torra and Narukawa [24, 25] proposed the hesitant fuzzy set (HFS) theory to deal with hesitation. HFS permits the membership degree to be represented by several values between 0 and 1. It is also an extensions of Zadeh' fuzzy set. After that, HFS has been applied in clustering analysis and decision-making [26-35]. For example, Xia and Xu [31] investigated the aggregation operators of HFSs and their applications in decision making. Chen [28] discussed the correlation coefficients of HFSs and applied them to deal with clustering analysis. Xu and Xia [33] presented the axiom definitions of distance and similarity measures for HFSs. They also proposed some distance of HFSs and obtained some similarity measures corresponding to the proposed distances of HFSs. It is worth noting that all of these definitions of distance and similarity measures did not take into account of the condition of triangle inequalities. However, we think that this condition is indispensable in the study of distance and similarity measures in that they are more consistent with humans' thinking. For example, the notions of fuzzy sets [7, 8], IFSs [18, 20, 36], IVFSs [26] and T2FSs [19] all satisfy the triangle inequalities. Therefore it is necessary to modify the axiom definitions of distance and similarity measures for HFSs. Furthermore, based on the new axiom definitions, we also propose some new distance and similarity measures between HFSs and apply them to multiple attribute hesitant fuzzy decision making.

The rest of this paper is organized as follows. In Section 2, we review some concepts of HFS and give the modified axiom definitions of distance and similarity measures between HFSs. Based on geometric distance model, in Section 3, we present some new geometric distance and similarity measures of HFSs Then we apply the proposed similarity measures of HFSs to decision-making in Section 4. We make the conclusions in Section 5.

II. PRELIMINARIES

Torra and Narukawa [24, 25] introduced the concept of HFS as follows.

Definition 2.1 Let X be a fixed set. An HFS on X is defined in terms of a function that when applied to X returns a subset of [0,1].

To be easily understood, Xia and Xu [33] express an HFS H as $H=\{\frac{h_H(x)}{x}|x\in X\}$, where $h_H(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x\in X$ to the set H. For convenience, $h=h_H(x)$ is called a hesitant fuzzy element (HFE), $n(h)$ denotes the number of values in h, and the values are arranged in decreasing order in this whole paper.

Xu and Xia [33] first gave the axiom definitions of distance and similarity measures between $HFSs$ as follows.

Definition 2.2 Let X be a fixed set, A and B be two HFSs over X. Then the distance of A and B is denoted by $D(\mathrm{A},\mathrm{B})$, which satisfies the following properties:

(1) $D(A,B)=D(B,A)$;

(2) $D(A,B)=0\Leftrightarrow A=B$;

(3) $0\le D(A,B)\le 1$.

Definition 2.3 Let X be a fixed set, A and B be two HFSs over X. Then the similarity measure of A and B is denoted by $\mathrm{s}(\mathrm{A},\mathrm{B})$, which satisfies the following properties:

(1) $S(A,B)=S(B,A)$;

(2) $S(A,B)=1\Leftrightarrow A=B$;

(3) $0\le S(A,B)\le 1$.

In many cases, however, $n(h_A(x))\ne n(h_B(x))$. When they are compared, we should extend the shorter one so that their length is the same. For instance, let $h_A(x)=\{0.7,0.5\}$, $h_B(x)=\{0.6,0.3,0.1\}$. Clearly, $n(h_A(x))<n(h_2(x))$, so $h_A(x)=\{0.7,0.5\}$ may be extended to $h_A(x)=\{0.7,0.5,0.5\}$ by adding the minimum value. In this paper, we all adopt the regulations.

Based on the above regulations, we define the following comparison laws.

Definition 2.4 Let X be a fixed set, A and B be two HFSs over X, $\mathrm{n_x}=\max\{\mathrm{n}(\mathrm{h_A}(\mathrm{x})),\mathrm{n}(\mathrm{h_B}(\mathrm{x}))\}$ for all x ∈ X. Then

(1) $h_A(x)$ is said to be inferior to $h_B(x)$, denoted by $h_A(x)\le h_B(x)$, if $h_A^{\sigma(i)}(x))\le h_B^{\sigma(i)}(x)$ for all $i=1,2,\cdots,n_x$. Especially, if $n_x=n(h_A(x))=n(h_B(x))$ and $h_A^{\sigma(i)}(x))\le h_B^{\sigma(i)}(x)$ for all $i=1,2,\cdots,n_x$, then $h_A(x)$ is said to be less than $h_B(x)$, denoted by $h_A(x)\le h_B(x)$.

(2) $h_A(x)$ is said to be equal to $h_B(x)$ if $h_A^{\sigma(i)}(x))=h_B^{\sigma(i)}(x)$ for all $i=1,2,\cdots,n_x$, denoted by $h_A(x)=h_B(x)$.

(3) HFS A is said to be an quasi subset of HFS B, denoted by $A\sqsubseteq B$, if $h_A(x)\preceq h_B(x)$ for all $x\in X$. Especially, if $h_A(x)\le h_B(x)$ for all $x\in X$, then A is called a subset of B, denoted by $A\subseteq B$.

(4) HFS A is said to be equal to HFS B, denoted by $A=B$, if $h_A(x)=h_B(x)$ for all $x\in X$.

Proposition 2.5 Let X be a fixed set, A and B be two $HFSs$ over X. If $\mathrm{h_A}(\mathrm{x})=\mathrm{h_B}(\mathrm{x})$, then $\mathrm{n}(\mathrm{h_A}(\mathrm{x}))=\mathrm{n}(\mathrm{h_B}(\mathrm{x}))$.

Proof. The proof is straightforward from Definition 2.4.

Based on Definition 2.4, we modify the axiom definitions of the distance and similarity measures as follows.

Definition 2.6 Let X be a fixed set, A, B and C be three HFSs over X, $D_{\max}=\max\{D(\mathrm{A},\mathrm{B})\}$. Then the distance of A and B is defined as $D(\mathrm{A},\mathrm{B})$, which satisfies the following properties:

$(D1)$ $0\le D(A,B)\le D_{max}$;

$(D2)$ $D(A,B)=0\Leftrightarrow A=B$;

$(D3)$ $D(A,B)=D(B,A)$;

$(D4)$ $A\sqsubseteq B\sqsubseteq C\Rightarrow D(A,B)\le D(A,C)$, $D(B,C)\le D(A,C)$.

If $(D1)$ is replaced by $(D1')$, then $D(A,B)$ is called a normalized distance, where $(D1')$ $0\le D(A,B)\le 1$.

Definition 2.7 Let X be a fixed set, A, B and C be three HFSs over X. Then the similarity measure between A and B is defined as $\mathrm{s}(\mathrm{A},\mathrm{B})$, which satisfies the following properties:

$(P1)$ $0\le S(A,B)\le 1$;

$(P2)$ $S(A,B)=1\Leftrightarrow A=B$;

$(P3)$ $S(A,B)=S(B,A)$;

$(P4)$ $A\sqsubseteq B\sqsubseteq C\Rightarrow S(A,C)\le S(A,B)$, $S(A,C)\le S(B,C)$.

III. SOME NEW SIMILARITY MEASURES FOR HFSS

In this section, we introduce some novel distance and similarity measures of HFSs.

Xu and Xia[33] introduced several geometric distance models of HFSs A and B on $X=\{x_1,x_2,\cdots,x_m\}$. Some of them are given as follows:

(1) Normalized hesitant fuzzy Hamming distance:

$$d_1(A,B)=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|\right) \quad (1)$$

(2) Normalized hesitant fuzzy Euclidean distance:

$$d_2(A,B) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^2\right)} \quad (2)$$

(3) Generalized normalized hesitant fuzzy distance:

$$d_3(A,B) = \left[\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)\right]^{1/p}, p>0. \quad (3)$$

Clearly, If $p=1$, then Eq. (3) is reduced to Eq. (1).

From Eq. (1), we know that

$$d_i = \frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)| \quad (4)$$

indicates the distance between the ith HFE of A and B, and $d_1(A,B)$ indicates the mean of distances between all elements of A and B.

Motivated by Eq. (4), we define another generalized normalized distance of A and B as:

$$d_4(A,B) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p>0. \quad (5)$$

which we call type-2 generalized normalized hesitant fuzzy distance. It is clear that Eq. (5) is different from Eq. (3). But if $p=1$, then Eq. (5) is also reduced to Eq. (1). If $p=2$, then Eq. (5) becomes type-2 normalized hesitant fuzzy Euclidean distance:

$$d_5(A,B) = \frac{1}{m}\sum_{i=1}^{m}\sqrt{\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^2}. \quad (6)$$

Theorem 3.1 $d_5(A,B)$ is a normalized hesitant fuzzy distance measure of $HFSs$ A and B.

Proof. It can be seen easily that $d_5(A,B)$ satisfies the properties $(D1')$, $(D2)$ and $(D3)$.So only prove the property $(D4)$. Let $A\sqsubseteq B\sqsubseteq C$, then $h_A(x_i)\le h_B(x_i)\le h_C(x_i)$ for each $x_i\in X$. It follows that $|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p \le |h_A^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)|^p$ and $|h_B^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)|^p \le |h_A^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)|^p$. Then

$$\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}\left|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)\right|^p \le \frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}\left|h_A^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)\right|^p,$$

$$\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_B^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)|^p \le \frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_C^{\sigma(j)}(x_i)|^p.$$

$$\Rightarrow d_4(A,B)\le d_4(A,C), d_4(B,C)\le d_4(A,C).$$

Thus the property $(D4)$ is obtained.

Based on Eq. (6), we further define some type-2 generalized hesitant distances as follows:

$$d_6(A,B) = \sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p>0. \quad (7)$$

$$d_7(A,B) = \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p>0. \quad (8)$$

$$d_8(A,B) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p>0. \quad (9)$$

Theorem 3.2 $d_i(A,B)(i=6,7,8)$ is a distance measure of $HFSs$ A and B, and satisfies the following properties:

(1) $0\le d_6(A,B)\le m$;

(2) $0\le d_7(A,B)\le \frac{1}{m}\sum_{i=1}^{m}(n_{x_i})^{1/p}$;

(3) $0\le d_8(A,B)\le \sum_{i=1}^{m}(n_{x_i})^{1/p}$.

Proof. The proof of the properties $(D2)-(D4)$ is similar to Theorem 3.1, We only prove $(1)-(3)$. Let $h_A^{\sigma(j)}(x_i)=1$ and $h_B^{\sigma(j)}(x_i)=0$ for all $x_i\in X$ and $j=1,2,\cdots,n_{x_i}$, then $d_5(A,B)=m$, $d_6(A,B)=\frac{1}{m}\sum_{i=1}^{m}(n_{x_i})^{1/p}$ and $d_7(A,B)=\sum_{i=1}^{m}(n_{x_i})^{1/p}$.

The L_P metric has been used to fuzzy sets and $IFSs$ [16]. Motivated by this means, the hesitant fuzzy L_P distance can be obtained:

$$d_9(A,B) = \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p\ge 1. \quad (10)$$

Clearly, if $p\ge 1$, then the type-2 generalized hesitant distance $d_7(A,B)$ becomes the hesitant fuzzy L_p distance $d_9(A,B)$.

However, there is an interesting result: if $p\to\infty$, then the hesitant fuzzy L_p distance $d_9(A,B)$ is reduced to normalized hesitant Hamming-Hausdorff distance

$$d_{10}(A,B) = \frac{1}{m}\sum_{i=1}^{m}\max_j|h_A^{\sigma(j)}(x_i)-h_B^{\sigma(j)}(x_i)|, \quad (11)$$

which has been defined by Xu and Xia [33].

To prove the above result, the following lemma is needed.

Lemma 3.3 Let $a_i\in\mathbb{R}$ and $a_i\ge 0, i=1,2,\cdots,k$. Then

$$\lim_{p\to\infty}(a_1^p+a_2^p+\cdots+a_k^p)^{\frac{1}{p}} = \max_i\{a_i\}, \quad p\ge 1.$$

Proof. Obviously whenever (i) $a_i = 0 (i = 1,2,\cdots,k)$, or (ii) $a_1 = a_2 = \cdots = a_k$, because $\lim_{p\to\infty} k^{1/p} = 1$. If $a_i \neq a_j, i \neq j, i,j = 1,2,\cdots,k$, then the following shows that

$$\lim_{p\to\infty}(a_1^p + a_2^p + \cdots + a_k^p)^{1/p} = \max_i\{a_i\}.$$

Without loss of generality, we suppose that $a_1 \geq a_2 \geq \cdots \geq a_k$, and let

$$y = (a_1^p + a_2^p + \cdots + a_k^p)^{1/p}.$$

Then

$$\lim_{p\to\infty} lny = \lim_{p\to\infty}\frac{a_1^p + a_2^p + \cdots + a_k^p}{p}.$$

Using L'Hospital's rule, we have

$$\begin{aligned}\lim_{p\to\infty} lny &= \lim_{p\to\infty}\frac{a_1^p \ln a_1 + a_2^p \ln a_2 + \cdots + a_k^p \ln a_k}{a_1^p + a_2^p + \cdots + a_k^p}\\ &= \lim_{p\to\infty}\frac{(\ln a_1 + (a_2/a_1)^p \ln a_2 + \cdots + (a_k/a_1)^p \ln a_k}{1 + (a_2/a_1)^p + \cdots + (a_k/a_1)^p}\\ &= lna_1.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{p\to\infty} y &= \lim_{p\to\infty}(a_1^p + a_2^p + \cdots + a_k^p)^{\frac{1}{p}}\\ &= a_1 = \max_i\{a_i\}.\end{aligned}$$

Theorem 3.4
$\lim_{p\to\infty} d_9(A,B) = \frac{1}{m}\sum_{i=1}^m \max_j |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|$.

Proof. It can be obtained directly from Lemma 3.3.

In many practical problems, however, the weight of the element $x_i \in X$ should be taken into account. Especially for MADM problems, the attributes usually are of different importance. Thus we need to consider the weighted distance of $HFSs$. Suppose that $w_i (i = 1,2,\cdots,m)$ is the weight of the element $x_i \in X$, $w_i \in [0,1]$ and $\sum_{i=1}^m w_i = 1$, then we obtain a type-2 generalized normalized hesitant fuzzy weighted distance

$$d_{11}(A,B) = \sum_{i=1}^m w_i \left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p > 0. \quad (12)$$

and a hesitant fuzzy L_p weighted distance

$$d_{12}(A,B) = \sum_{i=1}^m w_i \left(\sum_{j=1}^{n_{x_i}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p\right)^{1/p}, p \geq 1. \quad (13)$$

Obviously, if $w_i = 1/m, (i = 1,2,\cdots,n)$, then the Eq.s (12) and (13) are reduced to Eq.s (5) and (10), respectively.

It is found that the exponential operation is a very useful tool to deal with the similarity relation [6]. Therefore we adopt the exponential operation to a distance of $HFSs$ and get a new distance measure between $HFSs$. Let $d(A,B)$ be a distance of $HFSs$ A and B and $d_{max} = max\{d(A,B)\}$, then we define an exponential-type distance measure:

$$d_{13}(A,B) = \frac{1 - exp(-d(A,B))}{1 - exp(-d_{max})} \quad (14)$$

We give the following lemma to prove that Eq. (14) is a reasonable distance measure.

Lemma 3.5 Let $f(x) = \frac{1-\exp(-x)}{1-\exp(-m)}, x \in [0,m]$, then $f_{\min}(x) = f(0) = 0$ and $f_{\max}(x) = f(m) = 1$.

Proof. Since $f'(x) = \frac{exp(-x)}{1-exp(-m)} > 0, x \in [0,m]$, then $f(x)$ is increasing in $[0,m]$. Hence $f_{min}(x) = f(0) = 0$ and $f_{max}(x) = f(m) = 1$.

Theorem 3.6 Let $d(A,B)$ be a distance between HFSs A and B, and $d_{\max} = \max\{d(A,B)\}$. Then $d_{13}(A,B)$ is a normalized distance measure of HFSs A and B.

Proof. The properties $(D1') - (D3)$ is easily obtained, We only prove the property $(D4)$. Since $d(A,B)$ is a distance of $HFSs$ A and B, then $d(A,B) \leq d(A,C)$ and $d(B,C) \leq d(A,C)$ for $A \sqsubseteq B \sqsubseteq C$. By Lemma 3.5, we can obtain $d_{13}(A,B) \leq d_{13}(A,C)$ and $d_{13}(B,C) \leq d_{13}(A,C)$ for $A \sqsubseteq B \sqsubseteq C$.

From Theorem 3.6, we know that $d_{13}(A,B)$ is a normalized distance of $d(A,B)$, that is to say, we can use Eq. (14) to generate a normalized distance of $d(A,B)$.

We know that the similarity measure and distance are dual concepts. Hence we can use a distance to define a similarity measure.

Theorem 3.7 Let A and B be HFSs. Let f be a monotone decreasing function, d a distance and $d_{\max}$ the maximal distance. We define

$$s_0(A,B) = \frac{f(d(A,B)) - f(d_{max})}{f(0) - f(d_{max})}, \quad (15)$$

then $s_0(A,B)$ is a similarity measure of $HFSs$ A and B.

Proof. (1) Since f is a monotone decreasing function and $0 \leq d(A,B) \leq d_{max}$, then $f(d_{max}) \leq f(d(A,B)) \leq f(0)$. It follows that

$$0 \leq \frac{f(d(A,B)) - f(d_{max})}{f(0) - f(d_{max})} \leq 1.$$

(2) $d(A,B) = 0 \Leftrightarrow A = B$ implies $s_0(A,B) = 1 \Leftrightarrow A = B$.

(3) $d(A,B) = d(B,A)$ implies $s_0(A,B) = s_0(B,A)$.

(4) Let C be an HFS, and $A \sqsubseteq B \sqsubseteq C$. Since d is a distance, then $d(A,B) \leq d(A,C)$ and $d(B,C) \leq$

$d(A,C)$. Since f is a monotone decreasing function, then $f(d(A,C)) \leq f(d(A,B))$ and $f(d(A,C)) \leq f(d(B,C))$. These imply $s_0(A,C) \leq s_0(A,B)$ and $s_0(A,C) \leq s_0(B,C)$.

By Theorem 3.7, if we choose $f(x) = 1 - x$, then $s_0(A,B) = 1 - \frac{d(A,B)}{d_{max}}$. Based on Eq.s (3), (5) and (8), we obtain the corresponding similarity measures, respectively:

$$s_1(A,B) = 1 - \left[\frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p, \quad (16)$$

$$s_2(A,B) = 1 - \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p, \quad (17)$$

$$s_3(A,B) = 1 - \frac{1}{\sum_{i=1}^{m}(n_{x_i})^{1/p}}\sum_{i=1}^{m}\left(\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p. \quad (18)$$

where $p > 0$.

If we consider the weight of each element $x \in X$, then the weighted similarity measures can be obtained as follows:

$$s_4(A,B) = 1 - \left[\sum_{i=1}^{m} w_i\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p, \quad (19)$$

$$s_5(A,B) = 1 - \sum_{i=1}^{m} w_i\left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p, \quad (20)$$

$$s_6(A,B) = 1 - \frac{m}{\sum_{i=1}^{m}(n_{x_i})^{1/p}}\sum_{i=1}^{m} w_i\left(\sum_{j=1}^{n_{x_i}}|h_A^{\sigma(j)}(x_i) - hB\sigma(j)(xi)|p1/p. \quad (21)$$

where $p > 0$, $w_i \in [0,1]$ and $\sum_{i=1}^{m} w_i = 1$.

Especially, if $w_i = 1/m, (i = 1,2,\cdots,n)$, then the Eq.s (19), (20) and (21) are reduced to Eq.s (16), (17) and (18), respectively.

IV. AN APPLICATION IN MADM

In this section, we introduce an extended soft set model which is called multi-vague soft set by combining the multi-vague set and soft set. Some operations and their properties on multi-vague soft set will also be discussed.

In this section, we will apply the above proposed similarity measures to hesitant fuzzy MADM.

For a MADM problem, let $X = \{x_1, x_2, \cdots, x_m\}$ be a set of attributes, $H = \{H_1, h_2, \cdots, h_p\}$ a set of alternatives, and $w = \{w_1, w_2, \cdots, w_m\}^T$ the weight vector of attributes, where $w_i \in [0,1]$ and $\sum_{i=1}^{m} w_i = 1$.

Now we define respectively the positive ideal HFS and negative ideal HFS as follows:

$$H^+ = \{\frac{h_{H^+}(x_i)}{x_i}|x_i \in X\} \quad (22)$$

and

$$H^- = \{\frac{h_{H^-}(x_i)}{x_i}|x_i \in X\} \quad (23)$$

where

$$h_{H^+}(x_i) = \{h^{\sigma(k)}(x_i)|h^{\sigma(k)}(x_i) = \max_j\{h_{H_j}^{\sigma(k)}(x_i)\}, k = 1,2,\cdots,n_{x_i}\},$$

$$h_{H^-}(x_i) = \{h^{\sigma(k)}(x_i)|h^{\sigma(k)}(x_i) = \min_j\{h_{H_j}^{\sigma(k)}(x_i)\}, k = 1,2,\cdots,n_{x_i}\}.$$

Based on the aforementioned formulae of similarity measures between $HFSs$, we can calculate the similarity degree of the positive ideal HFS H^+ and alternative H_i, denoted by $s(H^+, H_i)$, and the similarity degree of the negative ideal HFS H^+ and alternative H_i, denoted by $s(H^-, H_i)$, respectively.

Then we define the relative similarity measure s_i for the alternative H_i as follows:

$$s_i = \frac{s(H^+, H_i)}{s(H^+, H_i) + s(H^-, H_i)}, i = 1,2,\cdots,m. \quad (24)$$

Obviously, the bigger the value s_i, the better the alternative H_i.

To illustrate the proposed similarity measures of $HFSs$ and the above approach of decision making, we present an example which is adapted from Example 1 in [31].

Example 4.1

It is very important that an appropriate energy policy is selected for affecting economic development of societies. Now, we suppose that the government will invest an energy project from five alternatives $H_i (i = 1,2,3,4,5)$. They are considered with four attributes (x_1: economic; x_2: environmental; x_3: technological; x_4: socio-political), the weight vector of which is $w = (0.15, 0.3, 0.2, 0.35)^T$. In order to obtain a more scientific and reasonable decision-making result, the government invite several decision makers to evaluate the performances of the five alternatives. For an alternative under an attribute, all possible evaluations provided by decision makers can be regarded as an HFE. For convenience, the all evaluated results are expressed by a hesitant fuzzy decision matrix, which is presented in TABLE I.

TABLE I
HESITANT FUZZY DECISION MAKING MATRIX

	x_1	x_2	x_3	x_4
H_1	{0.5,0.4,0.3}	{0.9,0.8,0.7,0.1}	{0.5,0.4,0.2}	{0.9,0.6,0.5,0.3}
H_2	{0.5,0.3}	{0.9,0.7,0.6,0.5,0.2}	{0.8,0.6,0.5,0.1}	{0.7,0.3,0.4}
H_3	{0.7,0.6}	{0.9,0.6}	{0.7,0.5,0.3}	{0.6,0.4}
H_4	{0.8,0.7,0.4,0.3}	{0.7,0.4,0.2}	{0.8,0.1}	{0.9,0.8,0.6}
H_5	0.9,0.7,0.6,0.3,0.1}	{0.8,0.7,0.6,0.4}	{0.9,0.8,0.7}	{0.9,0.7,0.6,0.3}

If we use the formulae of similarity measure $s_i(A,B)(i=4,5,6)$ to calculate the similarity degree of each alternative H_i and the positive ideal alternative H_i^+ (or negative ideal alternative H_i^-), then we get the rankings of these alternatives by Eq. (24). The results are gived in TABLE II-IV, respectively.

TABLE II
RESULTS OBTAINED BY THE SIMILARITY MEASURE $S_4(A,B)$

	H_1	H_2	H_3	H_4	H_5	Rankings
$p=1$	0.4719	0.47033	0.5111	0.47788	0.5547	$H_5 \succ H_3 \succ H_4 \succ H_1 \succ H_2$
$p=2$	0.46814	0.48052	0.5138	0.46197	0.55475	$H_5 \succ H_3 \succ H_2 \succ H_1 \succ H_4$
$p=6$	0.47238	0.48158	0.52557	0.4262	0.55783	$H_5 \succ H_3 \succ H_2 \succ H_1 \succ H_4$

TABLE III
RESULTS OBTAINED BY THE SIMILARITY MEASURE $S_5(A,B)$

	H_1	H_2	H_3	H_4	H_5	Rankings
$p=1$	0.4719	0.47033	0.5111	0.47788	0.5547	$H_5 \succ H_3 \succ H_4 \succ H_1 \succ H_2$
$p=2$	0.47016	0.46967	0.50993	0.48055	0.55334	$H_5 \succ H_3 \succ H_4 \succ H_1 \succ H_2$
$p=6$	0.47058	0.45747	0.51003	0.48376	0.54219	$H_5 \succ H_3 \succ H_4 \succ H_1 \succ H_2$
$p=10$	0.47124	0.4518	0.51049	0.48481	0.5389	$H_5 \succ H_3 \succ H_4 \succ H_1 \succ H_2$

TABLE IV
RESULTS OBTAINED BY THE SIMILARITY MEASURE $S_6(A,B)$

	H_1	H_2	H_3	H_4	H_5	Rankings
$p=1$	0.4728	0.4735	0.51883	0.4735	0.54951	$H_5 \succ H_3 \succ H_2 \succ H_4 \succ H_1$
$p=2$	0.46962	0.48329	0.51937	0.45865	0.55016	$H_5 \succ H_3 \succ H_2 \succ H_1 \succ H_4$
$p=6$	0.4976	0.49856	0.50208	0.4905	0.50783	$H_5 \succ H_3 \succ H_2 \succ H_1 \succ H_4$
$p=10$	0.49985	0.49978	0.50015	0.49819	0.50167	$H_5 \succ H_3 \succ H_1 \succ H_2 \succ H_4$

From TABLE II-IV, we find that $H_5 \succ H_3$ and they are superior to others whichever formula of similarity measure is used. And it is seen that the rankings are different when the different values of the parameter p, which can be regarded as the decision makers' risk attitude). Therefore, according to the decision makers' risk attitudes and actual situations, the proposed similarity measures can provide more choices for the decision makers. That is to say, the proposed methods are more flexible in practical application.

V. CONCLUSIONS

In this paper, we gave the definition of hesitant fuzzy quasi subset and presented the modified axiom definitions of distance and similarity measure of HFSs. Then, we proposed some novel hesitant distance measures based on the L_P metric, Euclidean distance, Hamming distance and exponential operation. We also investigated the relationships between distance and similarity measures. According to their relationships, some similarity measures between HFSs were obtained. Furthermore, we also applied the proposed similarity measures to a hesitant fuzzy MADM. The experiment results showed that the proposed similarity measures and approach were reasonable and efficient for hesitant fuzzy MADM.

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