International Journal of Engineering Trends and Technology
ISSN: 2231–5381 / https://doi.org/10.14445/22315381/IJETT-V74I1P112
Volume 74 Issue 1, 152-159, January 2026

Original Article

Coefficient Estimates of New Subclasses of Analytic Functions of Complex Order

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Received: 23 August 2025 Revised: 18 December 2025 Accepted: 25 December 2025 Published: 14 January 2026

Abstract *- The exploration of geometric characteristics and mapping of complex analytic functions has been actively developed since the late 19th century, leading to the field known as geometric function theory. One of the well-known problems in geometric function theory is the Fekete-Szegö problem, which seeks to determine the best possible bounds for certain functionals involving the differences of coefficients in the Taylor series expansion of analytic functions. This study aims to introduce new subclasses of analytic functions,* $\mathcal{M}_{q,\lambda}(\psi)$ *and* $\mathcal{L}_{q,\lambda}(\psi)$*, which are defined using the Sălăgeăn q-differential operator. This study also establishes the upper bounds on the Fekete-Szegö functional* $|a_3 - \eta a_2^2|$ *for functions in the new subclasses.*

Keywords *- Analytic functions, Fekete-Szegö functional, Geometric function theory, Quantum (or q-) calculus, q-Differential operator.*

1. Introduction

Geometric function theory emerged as an area of research at the end of 19^{th} and the beginning of 20^{th} centuries, primarily driven by the investigations of mathematicians like Henri Poincaré, Felix Klein, and Lars Ahlfors. The field aims to explore the geometric characteristics of complex analytic functions and their mapping. This theory has been instrumental in the study and analysis of the Fekete-Szegö problem [1].

The Fekete-Szegö problem, named after Hungarian mathematicians Lipót Fekete and Gábor Szegö, is a well-known problem in geometric function theory, originating in the early 20^{th} century [2]. First introduced by Lipót Fekete in 1922, the problem was further advanced by Gábor Szegö. The problem seeks to determine the best possible bounds for certain functionals involving the differences of coefficients in the Taylor series expansions of analytic functions. In particular, the Fekete-Szegö problem deals with the functional $|a_{\kappa+1} - a_\kappa|$, where a_κ represents the κ^{th} coefficient in the power series expansion of an analytic function.

The goal is to obtain the maximum value of $|a_{\kappa+1} - a_\kappa|$, referred to as the Fekete-Szegö constant, which serves as a measure of the geometric properties of the functions by quantifying the variation between consecutive coefficients. Furthermore, there are several studies about the coefficient estimates and their implications in different classes of complex functions, a crucial aspect of geometric function theory. For example, Aldweby and Darus [3] focused on the investigation of new classes of complex order q-starlike functions and q-convex functions, and the coefficient estimates associated with these classes. Additionally, the review also covers the Fekete-Szegö problem in relation to the q-derivative operator, examining its application within geometric function theory and complex analysis (see [4]). Besides, it discusses basic quantum (q-) calculus and fractional q-calculus operators, highlighting their significance in the research areas (see [5]). Moreover, the review also addresses Fekete-Szegö type problems and their applications, particularly focusing on subclasses of q-starlike functions involving symmetrical points (see [6]). Also, the research from Olatunji and Dutta [7] investigated the Fekete-Szegö problem for particular analytic functions defined by the q-derivative operator, considering symmetric and conjugate points, and also explored the problem for starlike functions of order β with respect to the generalized derivative operator (see [8]). In addition, Mohammed and Darus [9] studied the approximations and geometric properties of these q-operators within the optical disk. However, there has been a lack of attention towards subclasses of analytic functions involving the Sălăgeăn q-differential operator, especially analytic functions of complex order, and the associated Fekete-Szegö problems for these types of subclasses remain largely open.

An analytic function, also referred to as a holomorphic function, is a complex-valued function that exhibits differentiability at each point of a specific domain [10]. That is, a function $\hbar(\varepsilon)$ is known to be analytic in the open unit disk D if for each $\varepsilon \in D$, the derivative $\hbar'(\varepsilon)$ exists.

Consider $\mathcal{O}$ as the set of functions $\hbar$ analytic in D, defined as:

$$D = \{\varepsilon : \varepsilon \in \mathcal{C} \ \text{ and } \ |\varepsilon| < 1\}.$$

The Taylor series expansion of these functions is written as:

$$\hbar(\varepsilon) = \varepsilon + \sum_{\kappa=2}^{\infty} a_\kappa \varepsilon^\kappa, \qquad (1)$$

where $a_\kappa \in \mathcal{C}$ and $\kappa = 2, 3, \ldots$.

A univalent function, also known as a one-to-one function or injective function, is a complex-valued function that preserves distinctness [11]. In other words, for any distinct complex numbers ε_1 and ε_2 in the domain, their images under the function are also distinct. Formally, a function $\hbar: D \to \mathcal{C}$ is univalent if for all $\varepsilon_1, \varepsilon_2 \in D$, if $\varepsilon_1 \neq \varepsilon_2$, then $\hbar(\varepsilon_1) \neq \hbar(\varepsilon_2)$. Some examples of univalent functions can be found in [12, 13]. Thus, every univalent function is analytic, but not every analytic function is univalent.

In [14], they considered $\hbar(\varepsilon)$ and $\mathcal{H}(\varepsilon)$ to be analytic functions defined in D, with $\hbar(0) = \mathcal{H}(0)$. Assuming $\hbar$ is univalent, and the range of $\mathcal{H}$ lies within the range of $\hbar$, then

$$\omega(\varepsilon) = \hbar^{-1}(\mathcal{H}(\varepsilon))$$

is analytic in D, satisfies $\omega(0) = 0$, and has the property that $|\omega(\varepsilon)| < 1$ and is known as the Schwarz function. According to the Schwarz lemma in [14], except for the case where ω is simply a rotation of the disk, $|\omega(\varepsilon)| < |\varepsilon|$ for $0 < |\varepsilon| < 1$ unless. In most cases, the analytic function $\mathcal{H}$ is considered subordinate to the analytic function $\hbar$, denoted $\mathcal{H} \prec \hbar$ or $\mathcal{H}(\varepsilon) \prec \hbar(\varepsilon)$ if

$$\mathcal{H}(\varepsilon) = \hbar\big(\omega(\varepsilon)\big)$$

Where $|\varepsilon| < 1$ and $\omega(\varepsilon)$ is an analytic function satisfying $|\omega(\varepsilon)| < |\varepsilon|$. It should be noted that the superordinate function $\hbar$ does not necessarily have to be univalent.

Robertson [13] stated that D is referred to as starlike at the origin if $\omega \in D \Rightarrow [0, \omega] \subset D$ and an analytic function $\hbar$ is starlike in D if it conformally transforms D into a starlike domain such that $\hbar(0) = 0$, provided $\hbar'(0) \neq 0$. A function $\hbar \in \mathcal{O}$ is starlike if

$$Re\left(\frac{\varepsilon \hbar'(\varepsilon)}{\hbar(\varepsilon)}\right) \geq 0$$

for $\varepsilon \in D$, with the set of starlike functions represented by $\mathcal{S}^*$. If $\hbar \in \mathcal{O}$ is starlike of order ρ, it satisfies

$$Re\left(\frac{\varepsilon \hbar'(\varepsilon)}{\hbar(\varepsilon)}\right) \geq \rho$$

where $\rho > 0$, $\varepsilon \in D$, and this class represented by $\mathcal{S}^*(\rho)$.

Similarly, if $\omega, \omega' \in D \Rightarrow [\omega, \omega'] \subset D$, then the domain $D \subset \mathcal{C}$ is convex, and if an analytic function $\hbar$ maps D conformally, it is considered convex. A function $\hbar \in \mathcal{O}$ is convex if

$$Re\left(1 + \frac{\varepsilon \hbar''(\varepsilon)}{\hbar'(\varepsilon)}\right) \geq 0$$

for $\varepsilon \in D$, with the set of convex functions represented by $\mathcal{K}$. If $\hbar \in \mathcal{O}$ is convex of order ρ, it satisfies

$$Re\left(1 + \frac{\varepsilon \hbar''(\varepsilon)}{\hbar'(\varepsilon)}\right) \geq \rho$$

for $\rho > 0$, $\varepsilon \in D$, and this class represented by $\mathcal{K}(\rho)$. In [12], the function $\hbar$ in D is considered as an analytic function, and if there exists a convex or univalent function g, for $\varepsilon \in D$, then

$$Re\left(\frac{\hbar'(\varepsilon)}{g'(\varepsilon)}\right) > 0.$$

In terms of $\hbar(\varepsilon) \in \mathcal{O}$, the q-derivative of $\hbar(\varepsilon)$ in [15] is defined as

$$D_q \hbar(\varepsilon) = \frac{\hbar(q\varepsilon) - \hbar(\varepsilon)}{(1-q)\varepsilon}$$

where $q \neq 1$, $\varepsilon \neq 0$, $D_q \hbar(0) = \hbar'(0)$ provided $\hbar'(0)$ exists and $D_q^2 \hbar(\varepsilon) = D_q\left(D_q \hbar(\varepsilon)\right)$. Based on Equation 2, it can be inferred that

$$D_q \hbar(\varepsilon) = 1 + \sum_{\kappa=2}^{\infty} [\kappa]_q a_\kappa \varepsilon^{\kappa-1}$$

where $[\kappa]_q = \frac{1-q^\kappa}{1-q}$. To establish that $[\kappa]_q \to \kappa$ as $q \to 1^-$, let $g(\varepsilon) = \varepsilon^\kappa$. By applying Equation 2, then

$$D_q g(\varepsilon) = D_q(\varepsilon^\kappa) = \frac{1-q^\kappa}{1-q} \varepsilon^{\kappa-1} = [\kappa]_q \varepsilon^{\kappa-1}, \qquad (2)$$

and taking the limit as q approaches 1 from the left:

$$\lim_{q \to 1^-} D_q g(\varepsilon) = \lim_{q \to 1^-} [\kappa]_q \varepsilon^{\kappa-1} = \kappa \varepsilon^{\kappa-1} = g'(\varepsilon)$$

where g' denotes the classical derivative. Jackson *et al.* [16] also defined the q-integral as a mathematical concept that acts as a right inverse as follows:

$$\int_0^\varepsilon \hbar(t) D_q t = \varepsilon(1-q) \sum_{\kappa=0}^{\infty} q^\kappa \hbar(\varepsilon q^\kappa),$$

assuming the series converges.

In previous research, numerous novel subclasses of analytic functions using the q-derivative operator have been studied. Seoudy and Aouf [17] established two such subclasses, $\rho_q^*(\alpha)$ and $\varphi_q(\alpha)$, within the class $\mathcal{O}$ for $0 \leq \alpha < 1$. These subclasses are defined as follows:

$$\rho_q^*(\alpha) = \left\{ \hbar \in \mathcal{O} : Re\left(\frac{\varepsilon D_q \hbar(\varepsilon)}{\hbar(\varepsilon)}\right) > \alpha, \varepsilon \in D \right\},$$

and

$$\varphi_q(\alpha) = \left\{ \hbar \in \mathcal{O} : Re\left(\frac{D_q\left(\varepsilon D_q \hbar(\varepsilon)\right)}{D_q \hbar(\varepsilon)}\right) > \alpha, \varepsilon \in D \right\}.$$

For $\hbar \in \mathcal{O}$, the Sălăgeăn q-differential operator defined in [18] is written as

$$S_q^0 \hbar(\varepsilon) = \hbar(\varepsilon),$$

$$S_q^1 \hbar(\varepsilon) = \varepsilon D_q \hbar(\varepsilon),$$

$$S_q^m \hbar(\varepsilon) = \varepsilon D_q \left(S_q^{m-1} \hbar(\varepsilon)\right).$$

Equivalently, $S_q^m \hbar(\varepsilon)$ can be written in terms of $G_{q,m}(\varepsilon)$ as

$$S_q^m \hbar(\varepsilon) = \hbar(\varepsilon) \times G_{q,m}(\varepsilon) \tag{3}$$

for $\varepsilon \in D$, $m \in \mathcal{N} \cup \{0\} = \mathcal{N}_0$ and

$$G_{q,m}(\varepsilon) = \varepsilon + \sum_{\kappa=2}^{\infty} [\kappa]_q^m \varepsilon^\kappa. \tag{4}$$

By using Equations 3 and 4, the power series expansion of $S_q^m \hbar(\varepsilon)$ for a function $\hbar$ of the form given in Equation 1 is:

$$S_q^m \hbar(\varepsilon) = \varepsilon + \sum_{\kappa=2}^{\infty} [\kappa]_q^m a_\kappa \varepsilon^\kappa$$

where $\varepsilon \in D$, $\kappa = 2,3, \ldots$, $m \in \mathcal{N}_0$ and $[\kappa]_q^m = \left(\frac{1-q^\kappa}{1-q}\right)^m$.

Motivated by the existing literature, this study aims to establish new subclasses of analytic functions of complex order defined using Sălăgeăn q-differential operator and obtain the upper bound of the Fekete-Szegö functional $|a_3 - \eta a_2^2|$ for the functions in the classes.

2. Definition and Preliminary Results

This section presents the definitions of the new subclasses of analytic functions of complex order and provides the lemmas used in proofs of the main results. Denote $\mathcal{G}$ as the set of all analytic and univalent functions ψ in D, such that $\psi(0) = 1$, $Re\left(\psi(\varepsilon)\right) > 0$ and $\psi(\varepsilon)$ is a convex function.

Definition 1. A function $\hbar \in \mathcal{O}$ is considered belonging to the class $\mathcal{M}_{q,\lambda}(\psi)$ if it satisfies the subordination condition where

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{(1-\tau)S_q^m \hbar(\varepsilon) + \tau S_q^{m+1}\hbar(\varepsilon)} - 1\right) \prec \psi(\varepsilon)$$

where $0 \leq \tau < 1$, $m \in \mathcal{N}_0$, $\lambda \in \mathcal{C}\backslash\{0\}$ and $\psi \in \mathcal{G}$.

Remark 1.

(i) For $\lambda = 1$, $\tau = 0$ and $m = 0$, then

$$\mathcal{M}_{q,1}(\psi) = \mathcal{S}_q^*[A, B]$$

where the class $\mathcal{S}_q^*[A, B]$ was established in [19].

(ii) For $q \to 1^-$, $\lambda = 1$, $\tau = 0$ and $m = 0$, then

$$\lim_{q \to 1^-} \mathcal{M}_{q,1}(\psi) = \mathcal{S}^*(A, B)$$

where the class $\mathcal{S}^*(A, B)$ was established in [20].

(iii) For $\lambda = 1$, then

$$\mathcal{M}_{q,1}(\psi) = \mathcal{L}\textstyle\sum_q^k(\epsilon, \phi)$$

where the class $\mathcal{L}\sum_q^k(\epsilon, \phi)$ was introduced in [21].

Definition 2. A function $\hbar \in \mathcal{O}$ is considered belonging to the class $\mathcal{L}_{q,\lambda}(\psi)$ if it satisfies the subordination condition where

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{S_q^m \hbar(\varepsilon)} - 1\right) \prec \psi(\varepsilon)$$

where $m \in \mathcal{N}_0$, $\lambda \in \mathcal{C}\backslash\{0\}$ and $\psi \in \mathcal{G}$.

Remarks 2.

(i) For $q \to 1^-$, $\lambda = 1$ and $m = 0$, then

$$\lim_{q \to 1^-} \mathcal{L}_{q,1}(\psi) = \mathcal{S}^*(\phi)$$

where the class $\mathcal{S}^*(\phi)$ was introduced in [22].

Lemma 1. [22] For a function $t(\varepsilon) = 1 + c_1\varepsilon + c_2\varepsilon^2 + \cdots$ defined in D with a positive real part and for $\mu \in \mathcal{C}$, the following inequality holds:

$$|c_2 - \mu c_1^2| \leq 2\, max\{1; |2\mu - 1|\}.$$

The result is exact for

$$t(\varepsilon) = \frac{1+\varepsilon^2}{1-\varepsilon^2}$$

and

$$t(\varepsilon) = \frac{1+\varepsilon}{1-\varepsilon}.$$

Lemma 2. [22] For a function $t(\varepsilon) = 1 + c_1\varepsilon + c_2\varepsilon^2 + \cdots$ defined in D with a positive real part, the following inequality is achieved:

$$|c_2 - \mu c_1^2| \leq \begin{cases} 2(-2\mu+1), & if\ \mu \leq 0; \\ 2, & if\ 0 \leq \mu \leq 1; \\ 2(2\mu-1), & if\ \mu \geq 1. \end{cases}$$

where $\mu < 0$ or $\mu > 0$, the equality is achieved precisely when $t(\varepsilon) = \frac{1+\varepsilon}{1-\varepsilon}$ or any of its rotational equivalents. If $0 < \mu < 1$, then the equality is achieved precisely when $t(\varepsilon) = \frac{1+\varepsilon^2}{1-\varepsilon^2}$ or any of its rotational equivalents. If $\mu = 0$, the equality is achieved precisely when

$$t(\varepsilon) = \frac{(1+\varsigma)}{2}\frac{1+\varepsilon}{1-\varepsilon} + \frac{(1-\varsigma)}{2}\frac{1-\varepsilon}{1+\varepsilon}$$

or any of its rotational equivalents for $0 \leq \varsigma \leq 1$. If $\mu = 1$, the equality holds precisely when one of the functions has $t(\varepsilon)$ as its reciprocal that satisfies the equality condition for $\mu = 0$. Additionally, the upper bound mentioned is tight, and for $0 < \mu < 1$, it can be further tightened as follows:

$$|c_2 - \mu c_1^2| + \mu|c_1|^2 \leq 2$$

for $0 \leq \mu \leq \frac{1}{2}$ and

$$|c_2 - \mu c_1^2| + (1-\mu)|c_1|^2 \leq 2$$

for $\frac{1}{2} \leq \mu \leq 1$.

3. Main Results

For $0 \leq \tau < 1$, $m \in \mathcal{N}_0$, $\lambda \in \mathcal{C}\backslash\{0\}$ and $\psi \in \mathcal{G}$, the Fekete-Szegö functional for the class $\mathcal{M}_{q,\lambda}(\psi)$ is derived. Theorem 1. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 \neq 0$. If $\hbar$ is defined by Equation 1, and $\hbar \in \mathcal{M}_{q,\lambda}(\psi)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{[3]_q^m\left((1-\tau)q[2]_q\right)} max\{1; |Y|\}$$

where $Y = \frac{X_2}{X_1} + \frac{X_1\lambda}{(1-\tau)q}\left((\tau q+1) - [3]_q^m \cdot [2]_q^{1-2m}\eta\right)$.

Proof. If $\hbar \in \mathcal{M}_{q,\lambda}(\psi)$, then there exists $\omega(\varepsilon)$ satisfying

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{(1-\tau)S_q^m\hbar(\varepsilon) + \tau S_q^{m+1}\hbar(\varepsilon)} - 1\right) = \psi(\omega(\varepsilon)) \qquad (5)$$

where

$$S_q^m\hbar(\varepsilon) = \varepsilon + \sum_{\kappa=2}^{\infty}[\kappa]_q^m a_\kappa \varepsilon^\kappa$$

$$S_q^m\hbar(\varepsilon) = \varepsilon + [2]_q^m a_2\varepsilon^2 + [3]_q^m a_3\varepsilon^3 + \cdots \qquad (6)$$

and

$$S_q^{m+1}\hbar(\varepsilon) = \varepsilon + \sum_{\kappa=2}^{\infty}[\kappa]_q^{m+1} a_\kappa \varepsilon^\kappa$$

$$S_q^{m+1}\hbar(\varepsilon) = \varepsilon + [2]_q^{m+1} a_2\varepsilon^2 + [3]_q^{m+1} a_3\varepsilon^3 + \cdots \quad (7)$$

Substituting Equations 6 and 7 into Equation 5,

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{(1-\tau)S_q^m\hbar(\varepsilon) + \tau S_q^{m+1}\hbar(\varepsilon)} - 1\right)$$

$$= 1 + \left(\frac{[2]_q^m\left((1-\tau)q\right)}{\lambda}\right)a_2\varepsilon + B\varepsilon^2 + \cdots \qquad (8)$$

where

$$B = \left(\frac{[3]_q^m\left((1-\tau)q[2]_q\right)}{\lambda}\right)a_3 - \left(\frac{[2]_q^{2m}(\tau q+1)\left((1-\tau)q\right)}{\lambda}\right)a_2^2.$$

Let $t(\varepsilon)$ defined as

$$t(\varepsilon) = \frac{1+\omega(\varepsilon)}{1-\omega(\varepsilon)} = 1 + c_1\varepsilon + c_2\varepsilon^2 + \cdots. \qquad (9)$$

From Equation 9, $\omega(\varepsilon)$ can be computed as follows:

$$\omega(\varepsilon) = \frac{t(\varepsilon)-1}{t(\varepsilon)+1} = \frac{1}{2}\left(c_1\varepsilon + \left(c_2 - \frac{c_1^2}{2}\right)\varepsilon^2\right) + \cdots.$$

As $\omega(\varepsilon)$ is a Schwarz function, then $Re\left(t(\varepsilon)\right) > 0$ and $t(0) = 1$. Hence,

$$\psi(\omega(\varepsilon)) = 1 + \frac{X_1c_1}{2}\varepsilon + \left(\frac{X_1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\right)\varepsilon^2 + \cdots \qquad (10)$$

From Equation 5, the coefficients of ε and ε^2 Equations 8 and 10 can be compared, and obtain

$$a_2 = \frac{X_1c_1\lambda}{2[2]_q^m\left((1-\tau)q\right)} \qquad (11)$$

and

$$a_3 = \frac{X_1\lambda}{2[3]_q^m\left((1-\tau)q[2]_q\right)}V \qquad (12)$$

where $V = c_2 - \frac{1}{2}\left(1 - \frac{X_2}{X_1} - \frac{X_1\lambda(\tau q+1)}{(1-\tau)q}\right)c_1^2$. By using Equations 11 and 12, the value of $a_3 - \eta a_2^2$ can be computed for the equation as follows:

$$a_3 - \eta a_2^2 = \frac{X_1 \lambda}{2[3]_q^m\left((1-\tau)q[2]_q\right)}(c_2 - \mu c_1^2) \tag{13}$$

where

$$\mu = \frac{1}{2}\left(1 - \frac{X_2}{X_1} - \frac{X_1\lambda}{(1-\tau)q}\left((\tau q + 1) - [3]_q^m [2]_q^{1-2m}\eta\right)\right). \tag{14}$$

The result is obtained by applying Lemma 1. Hence, from Equation 13, then

$$a_3 - \eta a_2^2 = \left|\frac{X_1\lambda}{2[3]_q^m\left((1-\tau)q[2]_q\right)}\right| |c_2 - \mu c_1^2|$$

$$\leq \frac{|X_1\lambda|}{[3]_q^m\left((1-\tau)q[2]_q\right)} \max\{1; |2\mu - 1|\} \tag{15}$$

Substitute Equation 14 into Equation 15,

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{[3]_q^m\left((1-\tau)q[2]_q\right)} max\{1; |Y|\}$$

where $Y = \frac{X_2}{X_1} + \frac{X_1\lambda}{(1-\tau)q}\left((\tau q + 1) - [3]_q^m \cdot [2]_q^{1-2m}\eta\right)$. This concludes the justification of Theorem 1.

By setting $m = 0$ and $\tau = 0$ for Theorem 1, then

Corollary 1. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 \neq 0$. If $\hbar$ is defined by Equation 1, and $\hbar \in \mathcal{M}_{q,\lambda}(\psi)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{q[2]_q} \max\left\{\left\{1; \left|\frac{X_2}{X_1} + \frac{X_1\lambda}{q}\left(1 - [2]_q\eta\right)\right|\right\}\right.$$

By setting $m = 0$ and $\tau = 0$ in Theorem 1, a result is derived equivalent to Theorem 3.1 in [17].

By setting $m = 1$ and $\tau = 0$ for Theorem 1, then

Corollary 2. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 \neq 0$. If $\hbar$ is defined by Equation 1, and $\hbar \in \mathcal{M}_{q,\lambda}(\psi)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{q[2]_q[3]_q} \max\left\{1; \left|\frac{X_2}{X_1} + \frac{X_1\lambda}{q}\left(1 - \frac{[3]_q}{[2]_q}\eta\right)\right|\right\}.$$

Note that by setting $m = 1$ and $\tau = 0$ in Theorem 1, a result is derived equivalent to Theorem 3.2 in [17].

By setting $\lambda = 1$ for Theorem 1, then

Corollary 3. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 \neq 0$. If $\hbar$ is defined by Equation 1, and $\hbar \in \mathcal{M}_q(\psi)$ with $0 \leq \tau < 1$, then

$$|a_3 - \eta a_2^2| \leq \frac{|X_1|}{[3]_q^m\left((1-\tau)q[2]_q\right)} max\{1; |Z|\}$$

where $Z = \frac{X_2}{X_1} + \frac{X_1}{(1-\tau)q}\left((\tau q + 1) - [3]_q^m \cdot [2]_q^{1-2m}\eta\right)$.

The following theorem presents the Fekete-Szegö functional for the class $\mathcal{M}_{q,\lambda}(\psi)$ in Theorem 1 under different parameter cases.

Theorem 2. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 > 0$ and $X_2 \geq 0$. Let

$$\sigma_1 = \frac{(\tau q+1)X_1^2\lambda + (1-\tau)q(X_2 - X_1)}{[3]_q^m \cdot [2]_q^{1-2m} X_1^2 \lambda},$$

$$\sigma_2 = \frac{(\tau q+1)X_1^2\lambda + (1-\tau)q(X_2 + X_1)}{[3]_q^m \cdot [2]_q^{1-2m} X_1^2 \lambda},$$

and

$$\sigma_3 = \frac{(\tau q+1)X_1^2\lambda + (1-\tau)qX_2}{[3]_q^m \cdot [2]_q^{1-2m} X_1^2 \lambda}.$$

If $\hbar$ is given by Equation 1 in class $\mathcal{M}_{q,\lambda}(\psi)$ with $0 \leq \tau < 1$ and $\lambda > 0$, then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{X_2\lambda}{[3]_q^m(1-\tau)q[2]_q} + \frac{X_1^2\lambda^2}{[3]_q^m(1-\tau)^2q^2[2]_q}(\delta), & \eta \leq \sigma_1; \\ \frac{X_1\lambda}{[3]_q^m(1-\tau)q[2]_q}, & \sigma_1 \leq \eta \leq \sigma_2; \\ -\frac{X_2\lambda}{[3]_q^m(1-\tau)q[2]_q} - \frac{X_1^2\lambda^2}{[3]_q^m(1-\tau)^2q^2[2]_q}(\delta), & \eta \geq \sigma_2, \end{cases}$$

where $\delta = \tau q + 1 - [3]_q^m[2]_q^{1-2m}\eta$.

Proof. By applying Lemma 2,

Case 1, let $\eta \leq \sigma_1$

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{2[3]_q^m\left((1-\tau)q[2]_q\right)}(-4\mu + 2)$$

$$= \frac{X_2\lambda}{[3]_q^m(1-\tau)q[2]_q} + \frac{X_1^2\lambda^2}{[3]_q^m(1-\tau)^2q^2[2]_q}\left(\tau q + 1 - [3]_q^m[2]_q^{1-2m}\eta\right).$$

Case 2, let $\sigma_1 \leq \eta \leq \sigma_2$

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{2[3]_q^m\left((1-\tau)q[2]_q\right)}(2) = \frac{X_1\lambda}{[3]_q^m(1-\tau)q[2]_q}.$$

Case 3, let $\eta \geq \sigma_2$

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{2[3]_q^m\left((1-\tau)q[2]_q\right)}(4\mu - 2)$$

$$= -\frac{X_2\lambda}{[3]_q^m(1-\tau)q[2]_q} -$$

$$\frac{X_1^2\lambda^2}{[3]_q^m(1-\tau)^2q^2[2]_q}\left(\tau q + 1 - [3]_q^m[2]_q^{1-2m}\eta\right).$$

Furthermore, if $\sigma_1 \leq \eta \leq \sigma_3$

$$|a_3 - \eta a_2^2| + \frac{(1-\tau)q}{[3]_q^m[2]_q^{1-2m}X_1^2\lambda}(\mathscr{n})|a_2|^2 \leq \frac{X_1\lambda}{[2]_q[3]_q^m(1-\tau)q}$$

where $\mathscr{n} = X_1 - X_2 - \frac{X_1\lambda}{(1-\tau)q}\left(\tau q + 1 - [3]_q^m[2]_2^{1-2m}\eta\right)$ and if $\sigma_3 \leq \eta \leq \sigma_2$, then

$$|a_3 - \eta a_2^2| + \frac{(1-\tau)q}{[3]_q^m[2]_q^{1-2m}X_1^2\lambda}(\ell)|a_2|^2 \leq \frac{X_1\lambda}{[2]_q[3]_q^m(1-\tau)q}$$

where $\ell = X_1 + X_2 + \frac{X_1\lambda}{(1-\tau)q}\left(\tau q + 1 - [3]_q^m[2]_2^{1-2m}\eta\right)$.

By setting $\lambda = 1$ for Theorem 2, then

Corollary 4. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 > 0$ and $X_2 \geq 0$. Let

$$\sigma_1 = \frac{(\tau q+1)X_1^2 + (1-\tau)q(X_2 - X_1)}{[3]_q^m \cdot [2]_q^{1-2m}X_1^2},$$

$$\sigma_2 = \frac{(\tau q+1)X_1^2 + (1-\tau)q(X_2 + X_1)}{[3]_q^m \cdot [2]_q^{1-2m}X_1^2},$$

and

$$\sigma_3 = \frac{(\tau q+1)X_1^2 + (1-\tau)qX_2}{[3]_q^m \cdot [2]_q^{1-2m}X_1^2}.$$

If $\hbar$ is given by Equation 1 in class $\mathcal{M}_q(\psi)$ with $0 \leq \tau < 1$ and $\lambda > 0$, then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{X_2}{[3]_q^m(1-\tau)q[2]_q} + \frac{X_1^2}{[3]_q^m(1-\tau)^2q^2[2]_q}(\delta), & \eta \leq \sigma_1; \\ \frac{X_1}{[3]_q^m(1-\tau)q[2]_q}, & \sigma_1 \leq \eta \leq \sigma_2; \\ -\frac{X_2}{[3]_q^m(1-\tau)q[2]_q} - \frac{X_1^2}{[3]_q^m(1-\tau)^2q^2[2]_q}(\delta), & \eta \geq \sigma_2, \end{cases}$$

Where $\delta = \tau q + 1 - [3]_q^m[2]_q^{1-2m}\eta$.

Furthermore, if $\sigma_1 \leq \eta \leq \sigma_3$

$$|a_3 - \eta a_2^2| + \frac{(1-\tau)q}{[3]_q^m[2]_q^{1-2m}X_1^2}(\mathscr{f})|a_2|^2 \leq \frac{X_1}{[2]_q[3]_q^m(1-\tau)q}$$

where $\mathscr{f} = X_1 - X_2 - \frac{X_1}{(1-\tau)q}\left(\tau q + 1 - [3]_q^m[2]_2^{1-2m}\eta\right)$ and if $\sigma_3 \leq \eta \leq \sigma_2$, then

$$|a_3 - \eta a_2^2| + \frac{(1-\tau)q}{[3]_q^m[2]_q^{1-2m}X_1^2}(\mathscr{r})|a_2|^2 \leq \frac{X_1}{[2]_q[3]_q^m(1-\tau)q}$$

where $\mathscr{r} = X_1 + X_2 + \frac{X_1}{(1-\tau)q}\left(\tau q + 1 - [3]_q^m[2]_2^{1-2m}\eta\right)$.

The following theorem presents the result of the Fekete-Szegö functional for the class $\mathcal{L}_{q,\lambda}(\psi)$ where $m \in \mathcal{N}_0$, $\lambda \in \mathcal{C}\backslash\{0\}$ and $\psi \in \mathcal{G}$.

Theorem 3. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 \neq 0$. If $\hbar$ is defined by Equation 1, and $\hbar \in \mathcal{L}_{q,\lambda}(\psi)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|X_1\lambda|}{[3]_q^m([3]_q - 1)} max\{1; |L|\}$$

where $L = \frac{X_2}{X_1} + \frac{X_1\lambda}{[2]_q - 1}\left(1 - \frac{[3]_q^m([3]_q - 1)}{[2]_q^{2m}([2]_q - 1)}\eta\right)$.

Proof. If $\hbar \in \mathcal{L}_{q,\lambda}(\psi)$, then there exists $\omega(\varepsilon)$ satisfying

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{S_q^m\hbar(\varepsilon)} - 1\right) = \psi(\varepsilon) \quad (16)$$

Substituting Equation 6 and Equation 7 into Equation 16,

$$1 + \frac{1}{\lambda}\left(\frac{S_q^{m+1}\hbar(\varepsilon)}{S_q^m\hbar(\varepsilon)} - 1\right)$$

$$= 1 + \frac{1}{\lambda} \cdot \mathscr{d} \quad (17)$$

where $\mathscr{d} = \left([2]_q^{m+1} - [2]_q^m\right)a_2\varepsilon + \left(\left([3]_q^{m+1} - [3]_q^m\right)a_3 - \left([2]_q^{2m+1} - [2]_q^{2m}\right)a_2^2\right)\varepsilon^2 + \cdots$.

By using the same method as in Theorem 1, the coefficients of ε and ε^2 Equations 10 and 17 can be compared, and obtain

$$a_2 = \frac{X_1c_1\lambda}{2\left([2]^{m+1} - [2]_q^m\right)} \quad (18)$$

and

$$a_3 = \frac{X_1\lambda}{2[3]_q^{m+1} - [3]_q^m}. \quad (19)$$

By computing the value of $a_3 - \eta a_2^2$ and applying Lemma 1, then

$$|a_3 - \eta a_2^2| = \left|\frac{X_1\lambda}{2\left([3]_q^m([3]_q - 1)\right)}\right| |c_2 - \mu c_1^2|$$

$$\leq \frac{|X_1\lambda|}{[3]_q^m([3]_q - 1)} max\{1; |L|\}$$

where $L = \frac{X_2}{X_1} + \frac{X_1\lambda}{[2]_q - 1}\left(1 - \frac{[3]_q^m([3]_q - 1)}{[2]_q^{2m}([2]_q - 1)}\eta\right)$. This concludes the proof of Theorem 3.

The following theorem presents the Fekete-Szegö functional for the class $\mathcal{L}_{q,\lambda}(\psi)$ in Theorem 3 under different parameter cases.

Theorem 4. Given that $\psi(\varepsilon) = 1 + X_1\varepsilon + X_2\varepsilon^2 + \cdots$ with $X_1 > 0$ and $X_2 \geq 0$. Let

$$\sigma_4 = \frac{[2]_q^{2m}([2]_q-1)\left(\lambda X_1^2+([2]_q-1)(X_2-X_1)\right)}{([3]_q-1)\lambda X_1^2},$$

$$\sigma_5 = \frac{[2]_q^{2m}([2]_q-1)\left(\lambda X_1^2+([2]_q-1)(X_2+X_1)\right)}{([3]_q-1)\lambda X_1^2},$$

and

$$\sigma_6 = \frac{[2]_q^{2m}([2]_q-1)(\lambda X_1^2+([2]_q-1)X_2)}{([3]_q-1)\lambda X_1^2}.$$

If $\hbar$ is given by Equation 1 in class $\mathcal{L}_{q,\lambda}(\psi)$ with $\lambda > 0$, then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{X_2\lambda}{[3]_q^m([3]_q-1)} + \dfrac{X_1^2\lambda^2}{([2]_q-1)\left([3]_q^m([3]_q-1)\right)}(K), & \eta \leq \sigma_4; \\ \dfrac{X_1\lambda}{[3]_q^m([3]_q-1)}, & \sigma_4 \leq \eta \leq \sigma_5; \\ -\dfrac{X_2\lambda}{[3]_q^m([3]_q-1)} - \dfrac{X_1^2\lambda^2}{([2]_q-1)\left([3]_q^m([3]_q-1)\right)}(K), & \eta \geq \sigma_5, \end{cases}$$

where $K = 1 - \frac{[3]_q^m([3]_q-1)}{[2]_q^{2m}([2]_q-1)}\eta$.

Proof. The proof of Theorem 4 adopts the method of proof as in Theorem 2.

4. Conclusion

This study highlights the discovery of a new subclass within the analytic functions of complex order, specifically using the Sălăgeăn q-differential operator, $\mathcal{M}_{q,\lambda}(\psi)$ and $\mathcal{L}_{q,\lambda}(\psi)$. The results obtained contribute new insights to the Fekete-Szegö functionals for these classes of functions. These findings can be useful in areas such as complex dynamics and signal processing, where the application of the results may help with stability and approximation. This study may also serve as a stepping stone for future studies on subclasses involving q-based operators and applications in mathematics and applied science.

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