International Journal of Engineering Trends and Technology
ISSN: 2231–5381 / https://doi.org/10.14445/22315381/IJETT-V72I2P103

Volume 72 Issue 2, 19-28, February 2024

Original Article

Half-Sweep Newton-KSOR with Wave Variable Transformation to Solve Forced Porous Medium Equations

Graygorry Brayone Ekal[1], Jackel Vui Lung Chew[2], Khadizah Ghazali[3], Jumat Sulaiman[4]

[1,3,4]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah, Malaysia.*
[2]*Faculty of Computing and Informatics, Universiti Malaysia Sabah Labuan International Campus, Labuan F.T., Malaysia.*

[1]*Corresponding Author: graygorrybrayone@gmail.com*

Received: 29 June 2023 Revised: 04 August 2023 Accepted: 11 January 2024 Published: 03 February 2024

Abstract *- This paper intends to numerically solve the Forced Porous Medium Equation (FPME). The FPME is a Partial Differential Equation (PDE) that suffers from nonlinearity, making finding their exact solutions quite challenging. Therefore, approximate solutions to the FPME are necessary, leading to researchers' efforts to invent various analytical and numerical methods. Some existing numerical methods are sufficient to solve nonlinear PDEs. However, the computational costs are another issue to deal with. Hence, an alternative strategy to handle the nonlinearity and the computational costs is presented in this work. First, the FPME will be reduced into its corresponding travelling wave equation via a wave variable transformation. This reduction will significantly minimize the computational burden since the transformation results in an ordinary differential equation. Then, the reduced FPME will be discretized using the Half-Sweep central finite difference scheme to generate a system of nonlinear approximation equations. The Newton method addresses nonlinearity, producing a system of linear equations. Furthermore, the Kaudd Successive Over Relaxation (KSOR) iterative method is utilized to solve the linear system efficiently. Thus, with the combination of wave variable transformation, the Half-Sweep iteration technique, and the KSOR iterative method, the computational costs of finding approximate FPME solutions will be significantly minimized. This combination of techniques may be denoted as a Half-Sweep Newton-KSOR iterative method (HSN-KSOR). Finally, The efficiency of the HSN-KSOR iterative method in solving the FPME has been verified by numerical trials.*

Keywords *- Forced porous medium equation, Half-Sweep iteration, Kaudd Successive Over-relaxation, wave variable transformation.*

1. Introduction

The Forced Porous Medium Equation (FPME), sometimes called the porous medium equation with source terms, is defined as [1-3]

$$\frac{\partial u}{\partial t} = \rho \frac{\partial}{\partial t}\left(u^m \frac{\partial u}{\partial x}\right) + \mu u^r, \qquad (1)$$

Where ρ and μ are constants, while m and r are known rational numbers. The forcing term on the right-hand side of Equation 1, $f(u) = \mu u^r$, represents the effect of reaction or absorption [1], [4]. This equation arises in many areas of science for modelling the occurrences of certain physical phenomena. For instance, the gas or fluid flow in porous media modelling and to describe the spread of biological populations [5]. Furthermore, the FPME represents the unstable heat transport in the quiescent medium and population pressure in biological systems [6].

Additionally, Equation 1 is used to model the evolution of plasma temperature where the u term is set to be the temperature, $\frac{1}{\rho}$ as the particle density, and μu^r is the volumetric heating of plasma [7]. Moreover, the FPME may represent a wound-healing process and population dynamics models that are more realistic [8]. Then, the FPME is studied by [9] to model a gravity current, which is critical in observing the filtration laws. Understanding the filtration laws can help some industries tackle a membrane block and improve the performance of some membranes. In astronomy, the FPME is used to model the propagation of intergalactic civilization [10]. In recent years, some researchers have been interested in finding a solution to FPME and have proposed various methods, particularly in analytical or numerical approaches. For example, Pamuk [11] used the Adomian decomposition method (ADM) to solve the Porous Medium Equation (PME) with a source term or FPME.

The author developed a series of solutions for the PME without performing a linearisation and worrying about the number of nonlinear terms present in PME. Next, Biazar et al. [2] performed a homotopy perturbation method (HPM) on the FPME. They aimed to compare the PMES solutions derived from the HPM to those of the ADM and conclude that the solutions are equivalent. Furthermore, Sari [12] combined the sixth-order compact finite difference method with a low-storage total variation diminishing third-order Runge-Kutta to solve the FPME. The author took advantage of the tridiagonal band matrix produced by solving PDEs numerically, improving the proposed method's effectiveness. In addition, Nik et al. [13] developed a combination of HPM and He's polynomials to solve the FPME. The authors showed that the HPM could generate the approximate solution and He's polynomials simultaneously, making their method more efficient than ADM.

Moreover, Berx and Indekeu [14] proposed a beyond-linear use of the equation superposition (BLUES) iteration function method to solve some nonlinear PDEs, including the FPME. The authors concluded that the proposed method had a better convergence toward the approximate solution than ADM, the variational iteration method, and the variational iteration method with the Green function. Last but not least, Chew et al. [8] introduced a numerical technique called the quarter-sweep modified successive over-relaxation (QSMSOR) iterative approach to address the FPME. The performance of the proposed method is tested against the MSOR and the half-sweep MSOR iterative methods. The authors proved that their proposed method is efficient and accurate in solving the FPME compared to the controls.

The Porous Medium Equation (PME) and its complete form, i.e., the FPME, contributed enormously to developing the theories of nonlinear PDEs since its properties depart from the heat equation. In fact, the PME is sometimes called the nonlinear heat equation [1]. Motivated by the important applications of the FPME and the effort of finding the solution to the FPME, this paper intends to propose an effective and efficient numerical method to solve the FPME.

The method combines the wave variable transformation and the Half-Sweep Newton-Kaudd Successive Over Relaxation iterative method (HSN-KSOR). The wave variable transformation is one of the similarity techniques that can reduce a PDE into an ODE or to at least one independent variable from the original equation [15–17]. Hence, by employing this transformation in the proposed method, Equation 1 becomes simpler, minimizing the computation costs for its approximate solutions. On top of that, this strategy was highly inspired by the method proposed in previous studies, took advantage of the wave variable transformation to solve linear partial differential equations. However, their works only covered linear partial differential equations problems.

Hence, to fill the gap, this research extended their work to a nonlinear Partial Differential Equation (PDE) that suffers from nonlinearity. To add up, the transformed FPME is referred to as the travelling wave equation of FPME, and the theory, as well as the existence of travelling waves to FPME, have been discussed by researchers in [10, 16, 18]. Besides, authors also discuss travelling wave applications for other problems, which can be seen in the literature [15, 17, 19–24]. Additionally, the proposed method utilized the Half-Sweep iterative technique to accelerate the convergence of the approximation solutions for the FPME. The Half-Sweep iterative technique was first introduced by Abdullah in 1991 [25] to solve a Poisson problem. The idea of the Half-Sweep iteration concept is that the computation only takes half of the points in the solving domain of the problem. Several researchers have used this technique to solve problems, such as refer [26–27]. Lastly, the Newton and KSOR iterative methods were adopted to handle the nonlinearity in the FPME and solve the linear system, respectively.

The following sections discuss the development of the approximation equation for the FPME and the implementation of the HSN-KSOR iterative method (in Section 2). Then, the numerical results are presented in Section 3, and the conclusions of this work are given in Section 4.

2. Methodology

This section will derive the finite difference approximation equation for Equation 1 based on its corresponding travelling equation. The subsequent topic of discussion will be the formulation of the half-sweep Newton-KSOR iteration method.

2.1. Traveling Wave Equation

The wave variable transformation is one of the general similarity transformations that can reduce PDEs to ODEs, and it is given by [15], [17]

$$u(x,t) = u(z), \qquad z = x - ct, \tag{2}$$

and c being a constant representing the wave velocity. Therefore, utilizing Equation 2, the FPM is reduced into an ODE or its corresponding travelling wave equation by performing some chan rules on z. Hence, the following substitutions are made, which are:

$$\begin{cases} u_t = -cu_z \\ u_x = u_z \\ u_{xx} = u_{zz} \end{cases}. \tag{3}$$

Substituting Equation 3 into Equation 1 then gives the travelling wave equation for FPME of a form of

$$-c\frac{du}{dz} = \rho u^m \frac{d^2u}{dz^2} + amu^{m-1}\left(\frac{du}{dz}\right)^2 + \mu u^r, \tag{4}$$

Where $u = u(z)$, $-ct \le z \le 1 - ct$, and $t > 0$. One can see that Equation 4 only depends on one independent variable, z.

2.2. Half-Sweep Finite Difference Approximate Equation

Following this, the Half-Sweep finite difference approximation formulation for Equation 4 will be devised. To begin with, recall that the standard or full-sweep central finite difference scheme [28, 29] to discretize Equation 4 is given by

$$\begin{cases} u'(z_i) \approx \frac{u_{i+1}-u_{i-1}}{2h} \\ u''(z_i) \approx \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}, \end{cases} \tag{5}$$

for $i = 0,1,2,\cdots,(M-1)$. The h notation is $h = (z_1 - z_0)/M$ indicating the uniform step size with M being the subinterval in z direction. Then, substituting Equation 2 into Equation 4 yields

$$\alpha(u_{i+1} - u_{i-1}) = \beta u_i^m(u_{i+1} - 2u_i + u_{i-1}) + \gamma m u_i^m(u_{i+1} - u_{i-1})^2 + \mu u_i^r, \tag{6}$$

Where $\alpha = -c/2h$, $\beta = \rho/h^2$, and $\gamma = \rho/4h^2$. Fig. 1 shows the grid point distribution of the full-sweep iteration concept, where the approximate solutions will be computed using an iterative method at all the node points labelled by black dots.

Fig. 1 Finite grid points for full-sweep when $M = 8$

The half-sweep iteration approach only calculates the approximate solutions on the node locations indicated by black dots, specifically 2, 4, and 8 (refer to Fig. 2). Once the iterations are finished. The approximate solution is obtained for each black dot, and the white dots (1, 3, 5, and 7) are calculated using the direct method [30].

Fig. 2 Finite grid point for half-sweep when $M = 8$

Again, referring to Fig. 2 as a guide, the half-sweep central finite difference scheme can be expressed as

$$\begin{cases} u'(z_i) \approx \frac{u_{i+2}-u_{i-2}}{2H} \\ u''(z_i) \approx \frac{u_{i+2}-2u_i+u_{i-2}}{H^2}, \end{cases} \tag{7}$$

Where for $i = 2,4,6,\cdots,(M-4),(M-2)$. Here, the H notation is $H = 2h$, indicating the uniform step size in z direction. Thus, applying Equation 7 to Equation 4 yields.

$$\alpha(u_{i+2} - u_{i-2}) = \beta u_i^m(u_{i+2} - 2u_i + u_{i-2}) + \gamma m u_i^m(u_{i+2} - u_{i-2})^2 + \mu u_i^r, \tag{8}$$

Where $\alpha = -c/2H$, $\beta = \rho/H^2$, and $\gamma = \rho/4H^2$. As a result, Equation 8 is the half-sweep finite difference approximation equation for Equation 4.

2.3. Implementation of the Half-Sweep Newton-KSOR Iteration

Notice that Equation 8 may be expressed as a nonlinear system written as

$$f_i(u) = \alpha(u_{i+2} - u_{i-2}) - \beta u_i^m(u_{i+2} - 2u_i + u_{i-2}) - \gamma m u_i^m(u_{i+2} - u_{i-2})^2 - \mu u_i^r, \tag{9}$$

for $i = 2,4,6,\cdots,(M-2)$. Solving a nonlinear system is a tedious task. As a solution, Equation 9 is linearized using the Newton method [28], and the KSOR iterative method is then implemented to solve the resultant linear system. Besides, the linear system can be written as

$$\mathcal{J}\left(u^{(k)}\right)S^{(k)} = -f\left(u^{(k)}\right), \tag{10}$$

Where $u = (u_2, u_4, \cdots, u_{M-2})^T$, $S = (S_2, S_4, \cdots, S_{M-2})^T$, $f = (f_2, f_4, \cdots, f_{M-2})^T$, and k is the index of iteration.

Moreover, the Jacobian matrix in Equation 10 is defined as

$$\mathcal{J}\left(u^{(k)}\right) = \begin{pmatrix} \frac{\partial f_2(u)}{\partial u_2} & \frac{\partial f_2(u)}{\partial u_4} & \cdots & \frac{\partial f_2(u)}{\partial u_{M-2}} \\ \frac{\partial f_4(u)}{\partial u_2} & \frac{\partial f_4(u)}{\partial u_4} & \cdots & \frac{\partial f_4(u)}{\partial u_{M-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{M-2}(u)}{\partial u_2} & \frac{\partial f_{M-2}(u)}{\partial u_4} & \cdots & \frac{\partial f_{M-2}(u)}{\partial u_{M-2}} \end{pmatrix}^{(k)}_{\left(\frac{M}{2}-1\right)\times\left(\frac{M}{2}-1\right)} \tag{11}$$

To compute the approximate solutions $u^{(k)}$, the following formula is used:

$$u^{(k+1)} = S^{(k)} + u^{(k)}. \tag{12}$$

The system of linear equations stated in Equation 10 is very sparse and large with respect to the value of M. Hence, an efficient iterative method, namely the KSOR iterative method, will be used to solve the linear system. In addition, integrating the KSOR iterative method with the half-sweep iterative method elaborated in Section 2.2 will increase its efficiency. The KSOR iterative method was introduced by Youssef in 2012 as a new variant of the SOR iterative method.

It uses a less sensitive relaxation parameter, i.e., $\omega \in \mathbb{R} - [-2,0]$, than in the SOR method. Besides, the KSOR iterative method has the benefit of updating the initial component in the first equation at the first step, contributing to a swift convergence at the start. The iterative formula for the point KSOR, as derived from Equation 10, is provided as [31]

$$S_i^{(k+1)} = \frac{1}{1+\omega}\left[S_i^{(k)} + \frac{\omega}{\left(\frac{\partial f_i}{\partial u_i}\right)}\left((-f_i) - \sum_{j=2}^{i-2}\left(\frac{\partial f_i}{\partial u_j}\right)S_j^{(k+1)} - \sum_{j=i+2}^{M-2}\left(\frac{\partial f_i}{\partial u_j}\right)S_j^{(k)}\right)\right], \quad (13)$$

for $i = 2,4,\cdots,M-2$ and k is the iterative index. Therefore, based on equations (10) and (13), the general algorithm for the half-sweep Newton-KSOR iterative method to solve Equation 1 is presented in the following Algorithm 1.

Algorithm1: Half-Sweep Newton-KSOR iterative method

i. Set the value of $u^{(0)}$, tolerance error of u and S; $\varepsilon_u = 10^{-10}$, $\varepsilon_S = 10^{-10}$
ii. For $i = 2,4,\cdots,(M-2)$, do the following:
   a. Set $S^{(0)} = 0$
   b. Calculate $\mathcal{J}$ and $-\boldsymbol{f}$
   c. Calculate Equation 13 iteratively
iii. For $i = 1,3,\cdots,(M-1)$, compute directly.
iv. Check whether $\| \boldsymbol{S}^{(k+1)} - \boldsymbol{S}^{(k)} \| \leq \varepsilon_S$. If true, then proceed to step iv. Else, repeat step ii.
v. Calculate $\boldsymbol{u}^{(k+1)} = \boldsymbol{S}^{(k)} + \boldsymbol{u}^{(k)}$.
vi. Check whether $\| \boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)} \| \leq \varepsilon_u$. If true, then display approximate solutions, $\boldsymbol{u}$. Else, repeat step i until iv.

The optimal value of the relaxation parameter ω is obtained by repeatedly executing Algorithm 1 until the minimum number of iterations is reached.

3. Numerical Results

This section verifies the performance of half-sweep Newton-KSOR (HSN-KSOR) with some test problems. Three criteria are considered, i.e., the number of iterations until the approximate convergence solution is reached, the execution time (in seconds), and the maximum absolute error output. In addition, the Newton-Gauss Seidel (N-GS) and full-sweep Newton-KSOR (FSN-KSOR) iterative methods were set as benchmarks for the proposed iterative method. Regarding the tolerance error, $\varepsilon = 10^{-10}$ is used to determine the convergence of the solution at various grid sizes, i.e., $M = 256,512,1024,2048$, and 4096. The numerical computations were performed in C programming using a laptop computer with an Intel(R) Core i7-6500U CPU running at 2.60GHz and 8 GB of RAM. The following are the four selected FPME problems.

Example 1 [16]
Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right) + u^2, \quad (14)$$

Which is a quasilinear fast diffusion equation with a quadratic reaction term. A solution for this equation, which was provided by Polyanin and Zaitsev [16], was adopted for accuracy checking, namely

$$u(x,t) = \left(\frac{(x+C_1)^2}{2t} + C_2 t - 2tln|t|\right)^{-1}, \ t \neq 0,$$

Where C_1 and C_2 are arbitrary constants. In this implementation, C_1 and C_2 have been set to 0.35 and 1.35, respectively.

Example 2 [16]
Consider

$$\frac{\partial u}{\partial t} = \rho\frac{\partial}{\partial x}\left(u^3\frac{\partial u}{\partial x}\right) - \mu u, \quad (15)$$

the equation that describes the classical case of a gravity current in the air. Polyanin and Zaitsev [16] gave a particular solution for this equation, which is

$$u(x,t) = \left(Ae^{\frac{2bmt}{m+2}} - \frac{bm^2(x+B)^2}{2a(m+2)}\right)^{\frac{1}{m}},$$

Where A and B are arbitrary constants, for this work, A, ρ, B, and μ were set to be 1.35, 2, 0.35, and 1, respectively.

Example 3 [11]
Taking $m = 1$ and $r = 0$ in (1) then, it becomes

$$\frac{\partial u}{\partial t} = \rho\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \mu. \quad (16)$$

The exact solution for this equation is $u(x,t) = x + (\rho + \mu)t$, where a and b were set to be 1 and -3, respectively.

Example 4 [8]
Consider the equation representing a slow particle diffusion on a fresh membrane,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) - u. \quad (17)$$

The solution for this problem, provided by Chew et al., is

$$u(x,t) = \left(x - \frac{2}{\beta}\right)e^{\beta t} + \left(\frac{2}{\beta}\right)e^{2\beta t},$$

Where β is an arbitrary constant that has been set to -1.

The numerical results for each problem are presented in Tables 1 through 4. Meanwhile, Table 5 summarizes the numerical results for the test problems.

Furthermore, in addition, the results in Table 5 were computed by computing the following formula [25]:

$$\delta k\% = \left|\frac{k_{N-GS} - k_{HSN-KSOR}}{k_{N-GS}}\right| \times 100\%, \quad (18)$$

for the number of iterations while

$$\delta t\% = \left|\frac{t_{N-GS} - t_{HSN-KSOR}}{t_{N-GS}}\right| \times 100\%, \quad (19)$$

for the computational time.

Furthermore, the maximum absolute error is determined using [18]:

$$Max.Error = \max_{i\in[1,M-1]}\left\|u^{(k+1)} - u^{(k)}\right\|. \quad (20)$$

Table 1. Quantitative findings for Example 1

M	Method	*ω*	Iteration	Seconds	Max. Error
256	FSN-GS		165519	0.75	1.8667×10^{-04}
	FSN-KSOR	-2.0240	2975	0.02	1.8709×10^{-04}
	HSN-KSOR	-2.0481	1490	0.01	1.8651×10^{-04}
512	FSN-GS		547778	4.63	1.8766×10^{-04}
	FSN-KSOR	-2.0120	5912	0.09	1.8705×10^{-04}
	HSN-KSOR	-2.0240	2975	0.03	1.8668×10^{-04}
1024	FSN-GS		1730412	28.93	1.9423×10^{-04}
	FSN-KSOR	-2.0061	11667	0.30	1.8706×10^{-04}
	HSN-KSOR	-2.0120	5912	0.09	1.8672×10^{-04}
2048	FSN-GS		5121184	171.14	2.2096×10^{-04}
	FSN-KSOR	-2.0032	23221	1.15	1.8706×10^{-04}
	HSN-KSOR	-2.0061	11667	0.39	1.8673×10^{-04}
4096	FSN-GS		13829405	925.35	3.3397×10^{-04}
	FSN-KSOR	-2.0016	45132	4.48	1.8706×10^{-04}
	HSN-KSOR	-2.0031	22954	1.43	1.8673×10^{-04}

Table 2. Quantitative findings for Example 2

M	Method	*ω*	Iteration	Seconds	Max. Error
256	FSN-GS		289575	1.25	1.6586×10^{-03}
	FSN-KSOR	-2.0260	4346	0.03	1.6583×10^{-03}
	HSN-KSOR	-2.0522	2168	0.01	1.6579×10^{-03}
512	FSN-GS		992710	8.33	1.6599×10^{-03}
	FSN-KSOR	-2.0130	8630	0.12	1.6584×10^{-03}
	HSN-KSOR	-2.0260	4346	0.03	1.6583×10^{-03}
1024	FSN-GS		3319955	55.51	1.6644×10^{-03}
	FSN-KSOR	-2.0065	32772	0.85	1.6584×10^{-03}
	HSN-KSOR	-2.0130	8630	0.15	1.6584×10^{-03}
2048	FSN-GS		10749133	360.14	1.6880×10^{-03}
	FSN-KSOR	-2.0033	34071	1.71	1.6584×10^{-03}
	HSN-KSOR	-2.0065	17159	0.60	1.6584×10^{-03}
4096	FSN-GS		33849665	2263.30	1.7731×10^{-03}
	FSN-KSOR	-2.0017	68403	7.46	1.6584×10^{-03}
	HSN-KSOR	-2.0033	34071	2.32	1.6584×10^{-03}

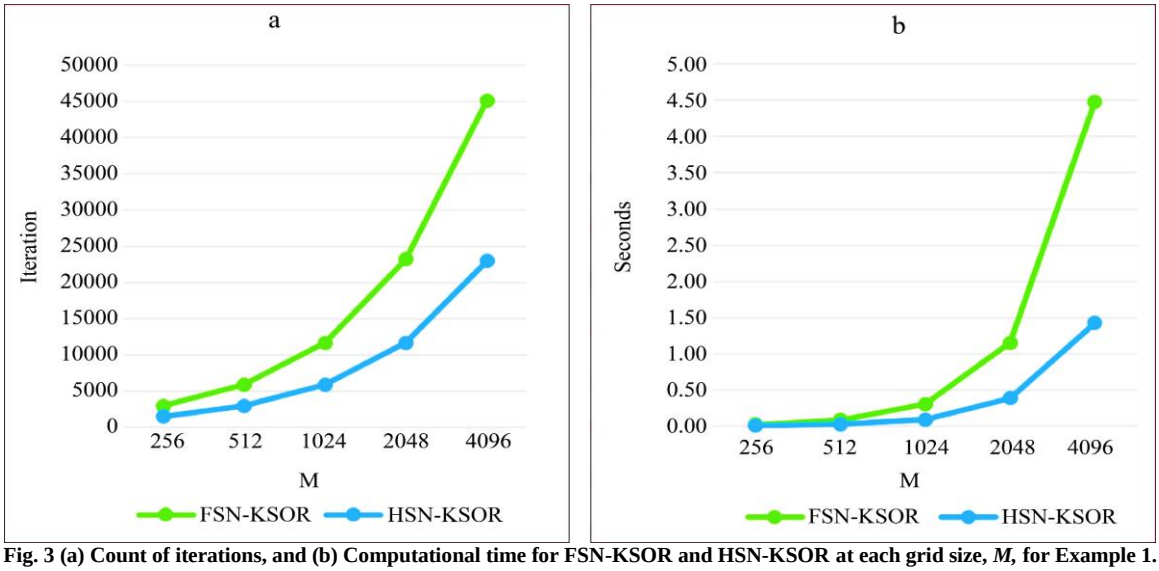

Fig. 3 (a) Count of iterations, and (b) Computational time for FSN-KSOR and HSN-KSOR at each grid size, *M*, for Example 1.

Table 3. Quantitative findings for Example 3

M	Method	*ω*	Iteration	Seconds	Max. Error
256	FSN-GS		228186	0.97	6.0643×10^{-07}
	FSN-KSOR	-2.0255	3224	0.02	1.5739×10^{-10}
	HSN-KSOR	-2.0510	1613	0.01	1.4048×10^{-10}
512	FSN-GS		779725	6.55	2.4306×10^{-06}
	FSN-KSOR	-2.0128	6424	0.09	1.7851×10^{-10}
	HSN-KSOR	-2.0255	3224	0.03	1.5739×10^{-10}
1024	FSN-GS		2599500	43.52	9.9409×10^{-06}
	FSN-KSOR	-2.0065	12782	0.33	2.3041×10^{-10}
	HSN-KSOR	-2.0128	6424	0.12	1.7851×10^{-10}
2048	FSN-GS		8416693	281.10	4.4445×10^{-05}
	FSN-KSOR	-2.0033	25327	1.28	3.0697×10^{-10}
	HSN-KSOR	-2.0065	12782	0.39	2.3041×10^{-10}
4096	FSN-GS		26872893	1802.07	1.5823×10^{-04}
	FSN-KSOR	-2.0017	50082	4.91	5.4396×10^{-10}
	HSN-KSOR	-2.0033	25327	1.75	3.0697×10^{-10}

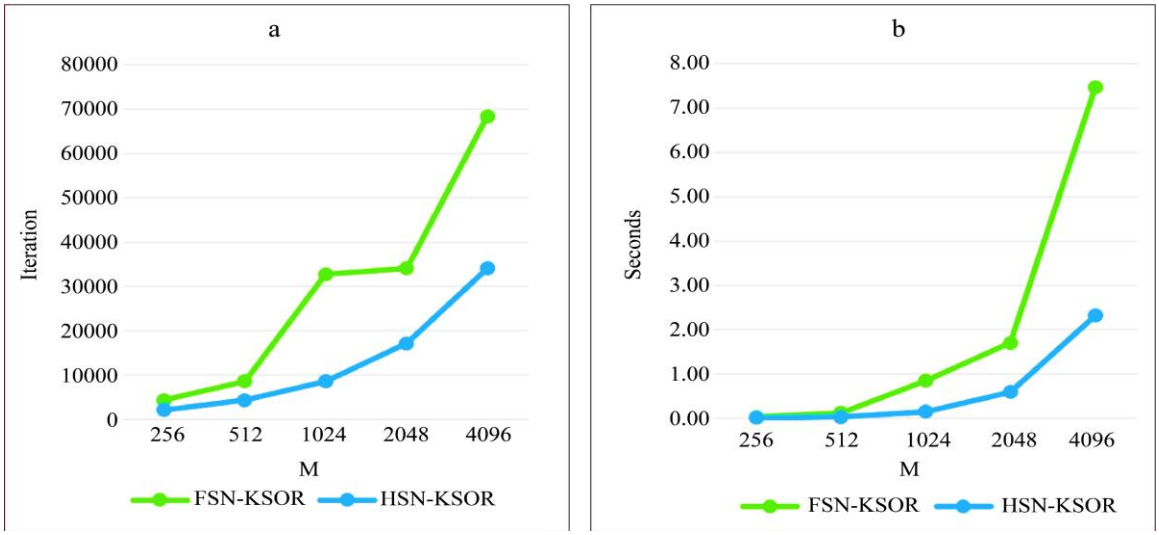

Fig. 4 (a) Count of iterations, and (b) Computational time for FSN-KSOR and HSN-KSOR at each grid size, M, for Example 2.

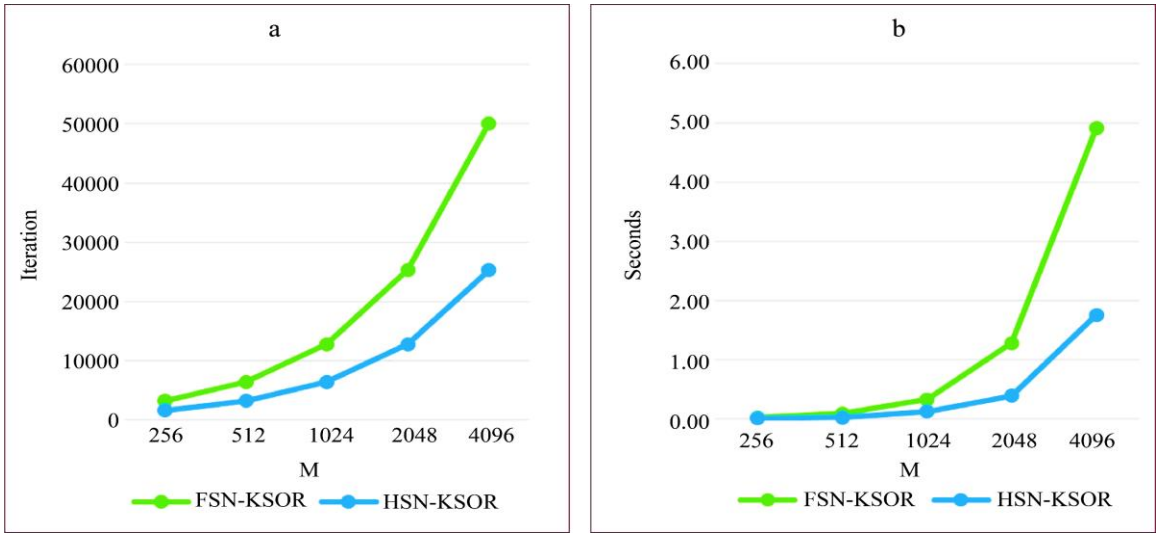

Fig. 5 (a) Count of iterations, and (b) Computational time for FSN-KSOR and HSN-KSOR at each grid size, M, for Example 3.

Table 4. Quantitative findings for Example 4

M	Method	*ω*	Iteration	Seconds	Max. Error
256	FSN-GS		189790	0.81	4.5050×10^{-03}
	FSN-KSOR	-2.0300	3433	0.03	4.5052×10^{-03}
	HSN-KSOR	-2.0608	1714	0.01	4.5052×10^{-03}
512	FSN-GS		649967	5.54	4.5056×10^{-03}
	FSN-KSOR	-2.0150	6847	0.10	4.5052×10^{-03}
	HSN-KSOR	-2.0300	3433	0.03	4.5052×10^{-03}
1024	FSN-GS		2217716	38.24	4.5093×10^{-03}
	FSN-KSOR	-2.0038	25621	0.65	4.5052×10^{-03}
	HSN-KSOR	-2.0150	6847	0.14	4.5052×10^{-03}
2048	FSN-GS		7382980	246.89	4.5241×10^{-03}
	FSN-KSOR	-2.0038	27093	1.35	4.5052×10^{-03}
	HSN-KSOR	-2.0075	13625	0.50	4.5052×10^{-03}
4096	FSN-GS		23636169	1583.74	4.5851×10^{-03}
	FSN-KSOR	-2.0020	53123	5.21	4.5052×10^{-03}
	HSN-KSOR	-2.0038	27093	1.69	4.5052×10^{-03}

Table 5. Percentage (%) improvement of HSN-KSOR against FSN-GS in terms of the number of iterations and computational time

	Iteration				
Example \ Grid size	256	512	1024	2048	4096
1	99.1	99.46	99.66	99.77	99.83
2	99.25	99.56	99.74	99.84	99.9
3	99.29	99.59	99.75	99.85	99.91
4	99.1	99.47	99.69	99.82	99.89
Computational time					
1	99.06	99.42	99.69	99.77	99.85
2	99.12	99.59	99.73	99.83	99.9
3	99.07	99.62	99.72	99.86	99.9
4	99.26	99.51	99.64	99.8	99.89

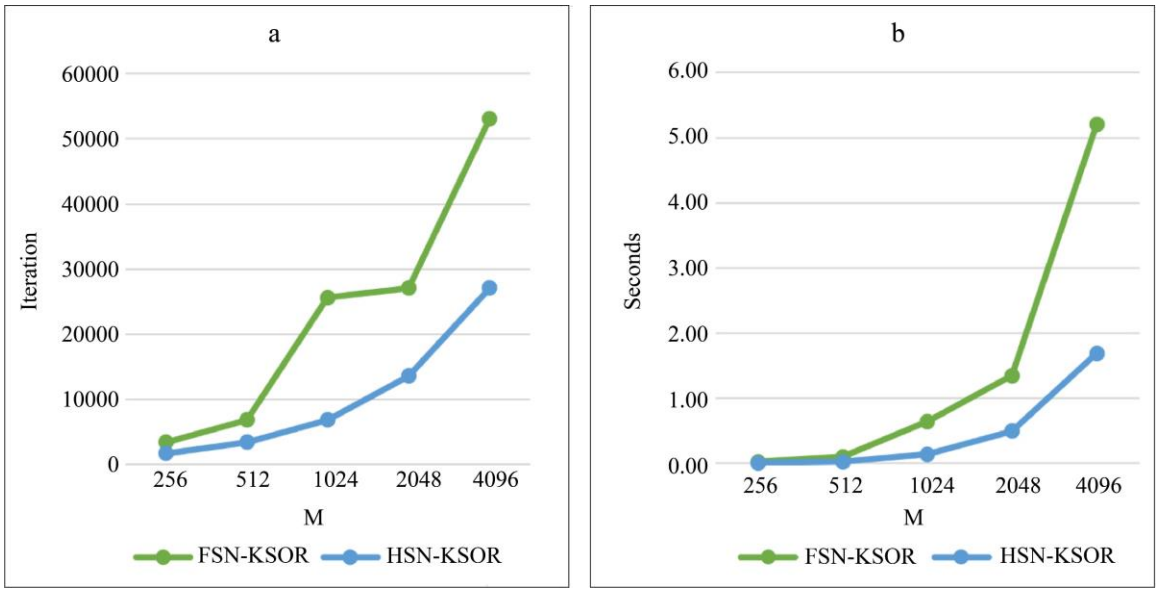

Fig. 6 (a) Count of iterations, and (b) Computational time for FSN-KSOR and HSN-KSOR at each grid size, M, for Example 4.

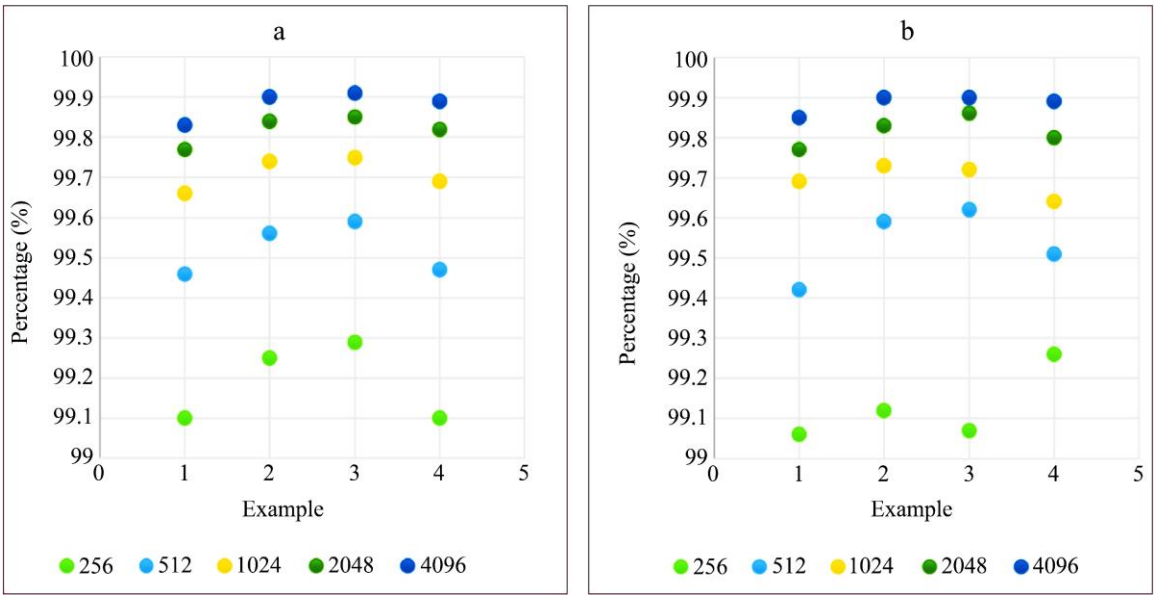

Fig. 7 Percentage improvement in (a) The number of iterations and (b) The computational time for Examples 1,2,3 and 4 for HSN-KSOR against FSN-GS at each grid size, *M*.

Based on Tables 1 through 4, it can be seen that the HSN-KSOR iterative method exhibited shorter processing time and required fewer iterations to achieve near-convergence solutions in comparison to the FSN-GS and FSN-KSOR iterative methods. The pattern was depicted in Figures 3 through 6, displaying both the iteration and time necessary to solve each problem across different grid sizes. Table 5 provides a concise overview of the improvements attained by the HSN-KSOR iterative method in comparison to the FSN-GS method, as seen in Figure 7.

Among four problems, the reduction in the number of iterations by the HSN-KSOR method against the FSN-GS method is at least 99.1 percent for a grid size of 256. For a grid size of 512, a minimum decrease of 99.46 percent in the iteration count has been obtained. In addition, given a grid size of 1024, a minimum of 99.66 percent of iterations experience a decrease across all problems. For a grid size of 2048, there was a minimum reduction of 99.77 percent across problems 1, 2, 3, and 4.

Finally, when the grid size was set to 4096, the reduction percentage achieved was at least 99.83% for all problems. Furthermore, the implementation of the HSN-KSOR iterative method significantly decreased the computing time produced by the FSN-GS by at least 99.06 percent while considering a grid size of 256. In addition, the computing time for all problems decreased by at least 99.42 percent for a grid size of 512. Additionally, a decrease of 99.64 percent was achieved with a grid size of 1024, while a reduction of 99.77 percent was achieved with a grid size of 2048. Finally, the HSN-KSOR iterative method shortened the computing time by a minimum of 99.85 percent for a specified grid size of 4096.

These events were made feasible by the ideal relaxation parameter value ω chosen in the implementation, shown in Tables 1 through 4, thereby speeding up the convergence to an approximation of the FPME solution. In addition, the application of half-sweep iteration helps to save computing expenses by calculating just half of the inner grid points inside a specific domain of interest. Besides, all methods show a good agreement in accuracy. However, the accuracy of the iterative methods HSN-KSOR and FSN-KSOR somewhat increases compared to FSN-GS.

4. Conclusion

In conclusion, the wave variable transformation and the Half-Sweep Newton-Kaudd SOR (HSN-KSOR) iterative method was successfully utilized to numerically solve the equations for porous media (FPME) equations. Adapting the wave variable transformation of the FPME before the discretization process significantly reduced the computational complexity since calculation on every time level is avoided.

Therefore, the performance of the proposed method is efficient and accurate in achieving an approximation of the FPME solution compared to the FSN-GS and FSN-KSOR. As a result, the suggested method offers a reasonable alternative numerical method compared to existing methods for approximating solutions for nonlinear parabolic partial differential equations that allow for travelling wave solutions.

Acknowledgements

The authors are grateful to Universiti Malaysia Sabah for the funding obtained upon publication of this research (GUG0624-1/2023).

References

[1] Juan Luis Vazquez, *The Porous Medium Equation*, Oxford University Press, pp. 1-183, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[2] Jafar Biazar, Zainab Ayati, and Hamideh Ebrahimi, "Homotopy Perturbation Method for General Form of Porous Medium Equation," *Journal of Porous Media*, vol. 12, no. 11, pp. 1121-1127, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[3] Yin Jing-Xue, Wang Liang-Wei, and Huang Rui, "Complexity of Asymptotic Behavior of Solutions for the Porous Medium Equation with Absorption," *Journal of Mathematical Sciences*, vol. 30, no. 6, pp. 1865-1880, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[4] Dmitry Fedchenko, and Nikolai Tarkhanov, "A Radó Theorem for the Porous Medium Equation," *Bulletin of the Mexican Mathematical Society*, vol. 24, pp. 427-437, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[5] F. Andreu et al., "Porous Medium Equation with Absorption and a Nonlinear Boundary Condition," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 49, no. 4, pp. 541-563, 2002. [CrossRef] [Google Scholar] [Publisher Link]

[6] Zhong Bo Fang, and Mei Wang, "Extinction Properties of Solutions for a Fast Diffusion Equation with Nonlocal Source," *Boundary Value Problems*, vol. 2013, pp. 1-15, 2013. [CrossRef] [Google Scholar] [Publisher link]

[7] Giulia Meglioli, and Fabio Punzo, "Blow-Up and Global Existence for Solutions to the Porous Medium Equation with Reaction and Slowly Decaying Density," *Journal of Differential Equations*, vol. 269, no. 10, pp. 8918-8958, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[8] J. V. L. Chew, J. Sulaiman, E. Aruchunan, and A. Sunarto, *Intelligent Systems Modeling and Simulation II: Efficient Iterative Approximation for Nonlinear Porous Medium Equation with Drainage Model*, Studies in Systems, Decision and Control. Cham, Switzerland: Springer, vol. 444, pp. 291–303, 2022. [CrossRef] [Google Scholar] [Publisher link]

[9] A.U. Krupp, I.M. Griffiths, and C.P. Please, "Inferring Filtration Laws from the Spreading of a Liquid Modelled by the Porous Medium Equation," *SIAM Journal on Applied Mathematics*, vol. 79, no. 4, pp. 1389-1404, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[10] Alejandro Gárriz, "Propagation of Solutions of the Porous Medium Equation with Reaction and their Travelling Wave Behaviour," *Nonlinear Analysis*, vol. 195, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[11] Nevin Pamuk, "Series Solution for Porous Medium Equation with a Source Term by Adomian's Decomposition Method," *Applied Mathematics and Computation*, vol. 178, no. 2, pp. 480-485, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[12] Murat Sari, "Solution of the Porous Media Equation by a Compact Finite Difference Method," *Mathematical Problems in Engineering*, vol. 2009, pp. 1-13, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[13] H. Saberinik et al., "Homotopy Perturbation Method and He's Polynomials for Solving the Porous Media Equation," *Computational Mathematics and Modeling*, vol. 24, no. 2, pp. 279-292, 2013. [CrossRef] [Google Scholar] [Publisher Link]

[14] Jonas Berx, and Joseph O. Indekeu, "BLUES Function Method Applied to Partial Differential Equations and Analytic Approximants for Interface Growth Under Shear," *Physical Review Research*, vol. 3, no. 3, pp. 1-17, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[15] Anna R. Ghazaryan, Stéphane Lafortune, and Vahagn Manukian, *Introduction to Traveling Waves*, CRC Press, pp. 1-174, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[16] Andrei D. Polyanin, and Valentin F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, 2nd ed., Taylor & Francis, pp. 1-1876, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[17] J. David Logan, *An Introduction to Nonlinear Partial Differential Equations*, 2nd ed., Hoboken, New Jersey: Wiley-Interscience, pp. 1-397, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[18] Tianyuan Xu, and Jingxue Yin, "Traveling Waves in Degenerate Diffusion Equations," *Communications in Mathematical Research*, vol. 39, no. 1, pp. 36-53, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[19] Sadaf Bibi, and Syed Tauseef Mohyud-Din, "Traveling Wave Solutions of KdVs Using Sine-Cosine Method," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, vol. 15, no. 1, pp. 90-93, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[20] Waipot Ngamsaad, and Kannika Khompurngson, "Self-Similar Solutions to a Density-Dependent Reaction-Diffusion Model," *Physical Review E*, vol. 85, no. 6, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[21] M.B.A. Mansour, "Travelling Wave Solutions for Doubly Degenerate Reaction–Diffusion Equations," *The ANZIAM Journal*, vol. 52, no. 1, pp. 101-109, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[22] M.B.A. Mansour, "Accurate Computation of Traveling Wave Solutions of Some Nonlinear Diffusion Equations," *Wave Motion*, vol. 44, no. 3, pp. 222-230, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[23] C. Atkinson, G.E.H. Reuter, and C.J. Ridler-Rowe, "Traveling Wave Solution for Some Nonlinear Diffusion Equations," *SIAM Journal on Mathematical Analysis*, vol. 12, no. 6, pp. 880-892, 1981. [CrossRef] [Google Scholar] [Publisher Link]

[24] Khalid K. Ali, and M.S. Mehanna, "Traveling Wave Solutions and Numerical Solutions of Gilson-Pickering Equation," *Results in Physics*, vol. 28, pp. 1-10, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[25] Abdul Rahman Abdullah, "The Four-Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver," *International Journal of Computer Mathematics*, vol. 38, no. 1-2, pp. 61-70, 1991. [CrossRef] [Google Scholar] [Publisher Link]

[26] Andang Sunarto, and Jumat Sulaiman, "Performance Numerical Method Half-Sweep Preconditioned Gauss-Seidel for Solving Fractional

Diffusion Equation," *Mathematical Modelling of Engineering Problems*, vol. 7, no. 2, pp. 201-204, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[27] Farhah Athirah Musli, and Azali Saudi, "Agent Navigation via Harmonic Potentials with Half-Sweep Kaudd Successive Over Relaxation (HSKSOR) Method," *IEEE 9th Symposium on Computer Applications & Industrial Electronics*, pp. 335-339, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[28] Richard L. Burden, J. Douglas Faires, and Annette M. Burden, *Numerical Analysis*, Ohio, United States of America: Cengage Learning, pp. 1-896, 2016. [Google Scholar] [Publisher Link]

[29] Xiaojia Yang, Yongbin Ge, and Lin Zhang, "A Class of High-Order Compact Difference Schemes for Solving the Burgers' Equations," *Applied Mathematics and Computation*, vol. 358, pp. 394-417, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[30] Ming-Ming Xu, Jumat Sulaiman, and Nur Afza Mat Ali, "Half-Sweep Refinement of SOR Iterative Method Via Linear Rational Finite Difference Approximation for Second-Order Linear Fredholm Integro-Differential Equations," *Mathematics and Statistics*, vol. 10, no. 3, pp. 486-497, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[31] I.K. Youssef, "On the Successive Overrelaxation Method," *Journal of Mathematics and Statistics*, vol. 8, no. 2, pp. 176-184, 2012. [CrossRef] [Google Scholar] [Publisher Link]