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Original Article

Piecewise Polynomial in solving Fredholm Integral Equation of Second Kind by using Successive over Relaxation method

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Abstract *- The Gauss-Seidel and Successive over Relaxation methods are two classic methods frequently used to solve the system linear equation. This study has stated that the Gauss-Seidel method was a linear solver, and both iterative methods operated in column and row spaces, respectively. In addition, the modification of the GS method has transformed the Successive Over Relaxation (SOR) thus, has improvised the iteration process in terms of iteration size and computational time. Therefore, to prove those parameters of the iteration process, the first-order piecewise polynomial has been applied to the Fredholm integral equation of second kind (FIE) with collocation point with composited trapezoidal quadrature method in order to get the approximation equations of Fredholm integral. This paper has successfully derived an approximation equation of composited trapezoidal method with first-order piecewise polynomial through the process of discretization on the FIEs with the consideration of vertex-centered type on the domain solutions. As mentioned previously, the numerical experiment has been tested on the approximation equation of composited trapezoidal method with first-order piecewise polynomial by using the derived algorithm of Gauss-Seidel (GS)* and *Successive Over Relaxation (SOR). The results that have also been recorded included the Maximum absolute error (MAE) besides the iteration size (IC) and computational time (CT).*

Keywords *- Collocation,* SOR*, GS, Piecewise, Polynomial, Trapezoidal.*

1. Introduction

The integral equation has been playing an important tool and is widely used in Mathematics, Physics, Thermodynamics and other fields. Many applications are widely applied in this study area, mainly in science and engineering. Moreover, there are good physical scenarios; thus, the integral equation is beneficial in applications. A few examples of application models that will be shown in this part were applied in the engineering field, such as medical image [1] in Fig. 1, wavelets [2] in Fig. 2, and mechanic quantum [3] in Fig. 3.

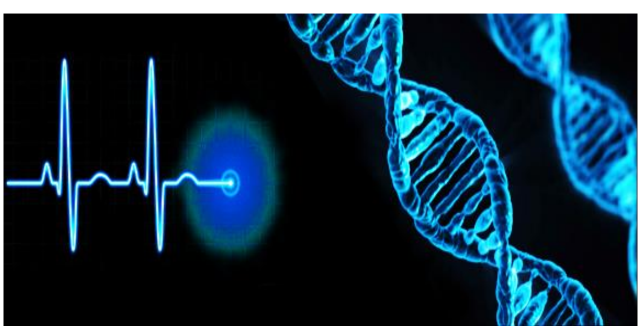

Fig. 1 The model of the medical image

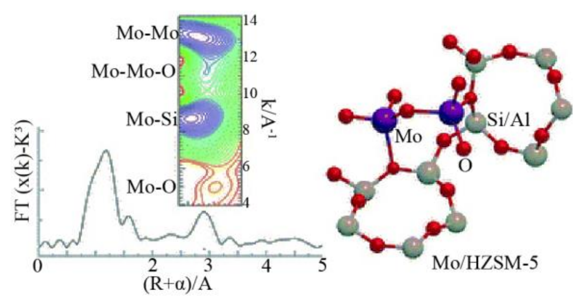

Fig. 2 The model of wavelets

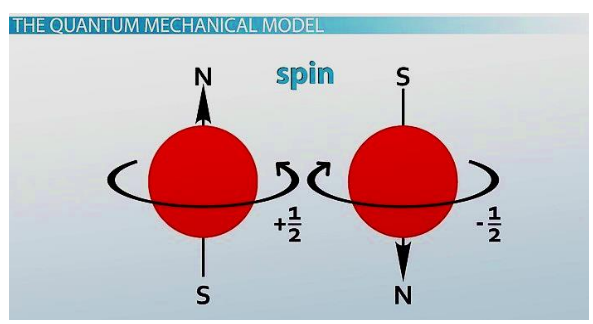

Fig. 3 The model of mechanic quantum

The integral equations consist of several types and cases, which can be Volterra's integral equation, Fredholm-volterra integral equation, Fredholm integral equation and many more [4,5,6]. A few kinds of Fredholm integral equations exist and are always found in any mathematic field, such as the Fredholm integral equation of the first kind, the Fredholm integral of the second kind and the Fredholm integral equation third kind.

The following is the equation of the first, second and third kinds of FIE(s), respectively.

$$\int_a^b k(x,t)U(t)dt = g(x), x \in [a,b], \quad (1)$$

$$U(x) + \lambda \int_a^b k(x,t)U(t)dt = g(x), x \in [a,b], \quad (2)$$

$$U(x) + \lambda \int_a^b k(x,t)U(t)dt = A(\mu)g(x), x \in [a,b], \quad (3)$$

However, this study only focuses on Fredholm's Integral equation (FIE) of the second kind. Referring to equation (2) briefly explains the characteristic of Fredholm integral equation of the second kind, where $k(x,t)$ is the kernel function of the integral equation, λ is the lambda parameter of the integral function. $U(x)$ is the unknown function and $U(t)$ is a known function. While a and b is the node point of interval on domain function [7, 8]. Based on the previous study, the solution of the integral equation is complicated and hard to solve by using the analytic solution. Thus the numerical method was introduced in solving the integral equation. Hence, this study is trying to solve the Fredholm integral equation using the numerical method.

The application that was invented initially came from expanding the ideas of integral equations itself. Moreover, there are many methods that can be used to get the approximation equations of integral equations, such as the Galerkin method, different finite methods, finite element method, Nystrom method, collocation method and others [9, 10, 11,22]. This research focuses intently on how the previous study applied the quadrature scheme and how it works since it was popular among researchers in solving integral equations. The quadrature scheme consists of several degrees of order, such as first-order, second-order, third-order, and to fifth-order. There were called as Trapezoidal method, 1/3 Simpson's method, 3/8 Simpson's method, Boole's method and Weddle's method, respectively. This quadrature scheme basically helps to improve the computational calculation method to get an excellent approximation solution [12].

Those characteristics within the quadrature method allow them to get excellent performance in the iteration process, time and maximum absolute error [13]. How to achieve efficient performance as an end approximation solution depends on the convergence rate of the iterative method picked in that study.

The introduction of polynomial piecewise also has inspired this research to increase the method's accuracy. The polynomial piecewise basically uses to find the approximate points within the domain solutions, where it calculates the coordinates on the corresponding curve.

Hence, based on what has been explained, this study is eager to determine the efficiency of the Fredholm integral equation of the second kind by generating the linear system equation through the discretization process using the first-order piecewise polynomial with collocation point with composited trapezoidal quadrature method. Then, formulate and carry out the research using the iterative method to get the approximate solutions on that generated linear system [15,30]. This will be discussed in the next part.

2. The Piecewise Polynomial Collocation on FIE second kind with full-sweep case

This section will review a few definitions of functions introduced in this research as this study applied the first-order piecewise polynomial with collocation with first-order quadrature on the Fredholm integral equation of the second kind. The definition of the quadrature method will be discussed in this part.

Before starting to derive an approximation equation of the proposed problem, its solution domain $I = [a,b]$ needs to be uniformly divided into n subintervals. Whereas all node points of x_i $i = 0,1,2,3,\ldots,n-1,n$ the solution domain can be stated as

$$a = x_0, x_1, x_2 < \cdots < x_{n-1} < x_n = b.$$

To construct a grid network of these node points, let us give the definition of $h$on the solution domain, which is called the length size of its subinterval on the interval $[\mathrm{a,b}]$. For nodes of equal step size, the length, h of subinterval of the $\mathrm{I} = [\mathrm{a,b}]$ can be given as

$$h = \frac{b-a}{n} \quad (4)$$

$U(x_0), U(x_1), \cdots, U(x_n)$ In this subsection, this study also discusses establishing the formulation of the first-order full-sweep quadrature method, namely full-sweep Composite Trapezoidal (FSCT), which is mainly applied to discretise an integral term of the problem (2). To do this matter, firstly, let us consider $\{x_0, x_1, x_2 < \cdots < x_{n-1} < x_n\}$ be $(n+1)$ real abscissas and $U(x_0), U(x_1), \cdots, U(x_n)$, represent their corresponding values, respectively. The grid network of these node points x_i $i = 0,1,2,3,\ldots,n-1,n$ of type can be illustrated in Fig. 4. As we derived the first-order piecewise polynomial collocation with first-order quadrature approximation equations of the problem (2), a set of the node points of type is the main part that needs to be highlighted as it makes the derivation of the first-order quadrature approximation equation is easier by identifying these node points.

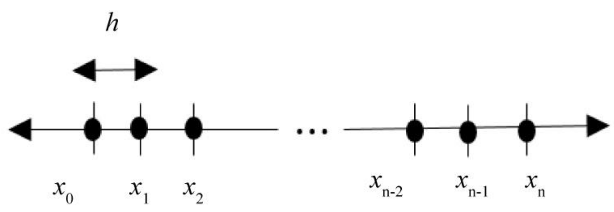

Fig. 4 Full-sweep interval domain of $I = [a, b]$ on Fredholm integral equation of the second kind

Referring to Fig. 4, the full-sweep grid network has been displayed all node points of type on the whole interval $[a, b]$ that have been treated as collocation node points for the full-sweep case. By applying the full-sweep case, consider the distribution of node points with $x_i = a + ih, i = 0,1,2, \cdots, n-1, n.$, with edge-vertex type.

To establish the formulation of the first-order quadrature method based on the collocation node points, let us consider Fig. 5, which shows a clear overview of the first-order quadrature method.

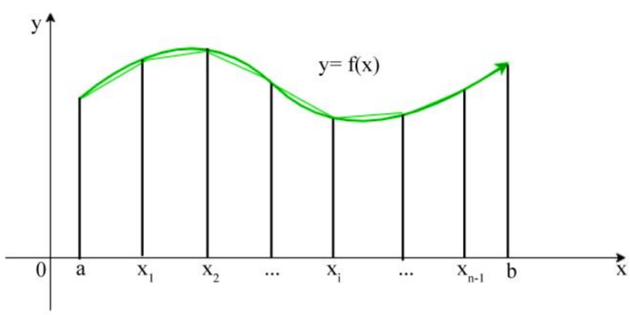

Fig. 5 The function of first-degree quadrature (Trapezoidal scheme)

Based on the collocation node points as depicted in Fig. 5, we attempt to present the FSCT scheme in which this idea of the FSCT scheme can be one of the Newton-Cotes types [16]. Using the same steps to get the formulation of the full-sweep Trapezoidal scheme being imposed to a function $f(x)$ at any subinterval $[x_i, x_{i+1}]$ can be stated as follow

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}[f_i + f_{i+1}], i = 0,1,2,3, \cdots n, \quad (5)$$

Generally, the formulation of the Full Trapezoidal schemes at any subinterval $[x_i, x_{i+\sigma}]$ can be stated as follows

$$\int_{x_i}^{x_{i+\sigma}} f(x)dx = \frac{(\sigma h)}{2}[f_i + f_{i+\sigma}], \quad (6)$$

$i = 0,1\sigma, \cdots, n-\sigma, n$

Where, the value of represents $\sigma = 1$ the full-sweep.

By referring to Fig. 5 and Eq. (5), it clearly,h is a subinterval of a finite sum for the solution domain, $[a, b]$ The following discussion is the expansion integration of the full-sweep Trapezoidal scheme (5) over the interval, $[a, b]$. Consequently, we can generalize to discretise an integral term of the problem (2) over the interval $[a, b]$ by using the full-sweep Composite Trapezoidal (FSCT) scheme into the finite sum of n as follows

$$\int_a^b f(x)dx = \int_{x0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \quad \dots (7)$$

Substitute Eq. (5) into Eq. (6), we get the FSCT scheme that can be rearranged to make it simpler as $\int_a^b f(x)dx = \frac{(h)}{2}(f_0 + f_1) + \frac{(h)}{2}(f_1 + f_2) + \cdots + \frac{(h)}{2}(f_{n-1} + f_n) \quad \dots (8)$

Then, Eq. (8) is formulated by using the finite sum of n

$$\therefore \int_a^b f(x)dx = \frac{(h)}{2}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right) \quad (9)$$

The recent section has wisely explained how the collocation method works on the Fredholm integral equation of the second kind. Systematically, the first-order piecewise polynomial approximation needs to be generated before applying the combination of the first-order quadrature method and the collocation scheme on the interval $I = [a, b]$. Let us start to define the first-order piecewise polynomial approximation function based on the full-sweep case over the solution domain, $[a, b]$ which can be stated as follows

$$U(x) = \sum_{i=0}^{n} H_i(x).\delta_i(x), x \in [a, b], \quad (10)$$

where the function of $H_i(x)$ and $\delta_i(x)$, $i = 0,1,2,3, \dots, n-1, n$ is specifically defined as

$$H_i(x) = \begin{cases} N_1(x)U_{i-1} + N_2(x)U_i, x_{i-1} < x < x_i \\ 0, otherwise \end{cases}$$

and

$$\delta_i(x) = \begin{cases} 1, x_{i-1} < x < x_i \\ 0, others \end{cases}$$

In fact, the function $H_i(x)$, $i = 0,1,2, \cdots$, n-1, n. is known as a piecewise linear function, and the function of $N_p(x)$, p = 1,2, is a linear function that can be defined as

$$N_1(x) = \frac{(x - x_{i-2})}{2h},$$

and

$$N_2(x) = \frac{(x_i - x)}{2h},$$

respectively.

Now to proceed with the process of discretization, substitute Eq. (10) into Eq. (2) as we want to construct the corresponding approximation equation as follows:

$$U(x) + \lambda \int_a^b k(x,t) \sum_{j=0}^{n} H_j(x).\delta_i(x)dt = g(x), \quad (11)$$

To make it simple, we simplify Eq. (11), which can easily be understood as

$$U(x) + \lambda \sum_{j=0}^{n} \int_a^b k(x,t).H_j(x)\delta_i(x)dt = g(x), \quad (12)$$

As explained in the previous part, the collocation scheme is one of the important processes in getting the approximation equation. Now, impose all the collocation node points by using the edge-vertex approach into Eq. (12) to get

$$U_i + \lambda \sum_{j=0}^{n} H_{ji} \int_a^b k(x_i,t).\delta_{ji} dt = g_{i_i} \quad (13)$$

for $i = 0,1,2,\cdots,n-1,n.$,
where
$U_i = U(x_i),\ H_j(x_i) = H_{ji,}\ \delta_j(x_i) = \delta_{ji}, g(x_i) = g_i.$

By referring to Eq. (5), start applying the first-order quadrature method, namely FSCT, in the discretization process of an integral term in Eq. (13). Before applying the FSCT scheme, let us rewrite Eq. (5) in the following equation

$$\int_a^b R(t)dt = \sum_{j=0}^{n} A_j R(t_j) + \varepsilon_n(R) \quad (14)$$

Where $A_j, j = 1,2,3,4,\ldots n.$ are the quadrature weights of the quadrature method and $t_i = a + ih,\ i = 0,1,2,\ldots,n-1,n.$ is pointed as a coordinate of partition over the interval integration of the $[a,b]$. Below is the quadrature weight function A_j that does not rely on the function of $R(t)$ and satisfies the following relation

$$A_j = \begin{cases} h, j = 0, n \\ 2h, others \end{cases} \quad (15)$$

By substituting Eq. (15) into Eq. (13), the general form of the FSCT piecewise linear collocation approximation equation to approximate the problem (2) can be constructed as

$$U_i + \sum_{j=0}^{n} A_j G_{ji} U_j = g_i \quad (16)$$

for $i = 0,1,2,\ldots,n-1,n.$,
where
$k_{i,j} = k(x_i,t_j)$ and $G_{i,j} = \lambda \int_{x_{i-1}}^{x_i} k(x_i,t) H_j(x_i).\delta_j(x_i)\, dt.$

Regarding the truncation error of , it has no problem not to pay attention to the truncation error to get the dense and linear system generated by the first-order polynomial piecewise collocation with first-order quadrature approximation equations. The development of the linear system has been generated after the substitution of Eq. (15) into Eq. (16) as follows

$$F\underline{U} = \underline{g} \quad (17)$$

where,

$$F = \begin{bmatrix} 1+G(x_0,0) & G(x_0,1) & \cdots & G(x_0,n) \\ G(x_1,0) & 1+G(x_1,1) & \cdots & G(x_1,n) \\ \vdots & \vdots & \ddots & \vdots \\ G(x_n,0) & G(x_n,1) & \cdots & 1+G(x_n,n) \end{bmatrix}_{(n\times n)},$$

$$\underline{U} = [U_0 \quad U_1 \quad \cdots \quad U_n],$$

$$\underline{g} = [g_0 \quad g_1 \quad \cdots \quad g_n].$$

Clearly, it can be found that the key feature of the coefficient matrix F of the linear system (17) is a large-scale and dense matrix. However, applying the first-order piecewise polynomial with the collocation scheme, the key feature of its dense coefficient matrix is due to the FSCT discretization scheme for the integral term of the problem (2).

3. Computational Experiments on Fredholm Equation of Second Kind

3.1. Iterative Method

According to the linear system (17), clearly, this study needs to solve the linear system iteratively to get its approximation solution. This paper uses the Gauss-Seidel (GS) as the control method. The Gauss-Seidel method is one of the classic iterative methods often used, as the algorithm itself was inspired by the Jacobi iterative method. However, the Gauss-Seidel method performs better than the Jacobi method but is slower than the Successive over Relaxation method (SOR) but has difficulty finding the optimal relaxation parameter since it has expensive computational calculations [17].

The derivation of Gauss-Seidel will be shown as follows: Referring to Eq. (17), consider the linear system of $F\underline{U} = \underline{g}$, $F \in Max_{nxn,} b \in \Re^n$, and let

$$F = D - L - V \quad (18)$$

as decomposition of F, where Dis the diagonal matrix, L is lower triangular and V is the upper triangular matrix.

The vector form of Gauss-Seidel is generated in Eq. (19) by applying the decomposition of Eq. (18) into the linear system of Eq. (17)

$$\underline{U}^{(k+1)} = (D-L)^{-1} V \underline{U}^{(k)} + (D-L)^{-1} \underline{g} \quad (19)$$

In order to get the SOR vector form, consider the relaxation factor of parameter ω [18].

The derivation can be done by taking into account the factor by applying into Eq. (17) and rewrite as

$$\omega F \underline{U} = \omega \underline{g} \quad (20)$$

Note that the parameter's range of SOR to get the optimal value is [1,2).

The formula of the SOR iterative method is stated in vector form as follows:

$$\underline{U}^{(k+1)} = (D-\omega L)^{-1}[(1-\omega)D + \omega \underline{V}]\underline{U}^{(k)} + \omega(D - \omega L)^{-1}\underline{g} \quad \ldots (21)$$

In the process of numerical experiment, when the parameter's value is 1 or $\omega = 1$, the result is that printed out was for the Gauss-Seidel method value. As to improve the approximation solution, the range of the relaxation factor must be followed within the stated range. Thus, it can increase the performance of the numerical experiment of SOR iteration.

4. Algorithms of the SOR Iterative Method

i. Set up the initial value of $\underline{U}^{(0)} \leftarrow 0, \varepsilon \leftarrow 10^{-10}$. . Assign the optimum value of ω.

ii. For $k = 0,1,2,\cdots n$, execute

 a. For $i = 0,1,\cdots,n$, calculate $\underline{U}^{(k+1)} = (D-\omega L)^{-1}[(1-\omega)D + \omega\underline{V}]\underline{U}^{(k)} + \omega(D-\omega L)^{-1}\underline{g}$ Examine the convergence test

$$|U_i^{(k+1)} - U_i^{(k)}| \leq \varepsilon = 10^{-10}$$

 If it fits the convergence test, proceed to step iii. Otherwise, redo step ii.

iii. Show the approximate solutions of $|U_i^{(k+1)}$.

5. Results and Discussion

5.1. Computational Experiment

This section will portray how the derived algorithms of GS and SOR were implemented on the Fredholm integral equation of the second kind. To run the computational calculation, need to consider a few parameters, which are the iteration size (IC), computational time (CT) and maximum absolute error (MAE) [14,19,20]. We chose five size grid $N = 256,512,1024,2048,4096$

The following equations are three selected examples in this paper:

Example 1 [21]:

$$y(x) = x + \int_0^1 4xt - x^2 y(t)dt \quad (22)$$

Exact solution of (22) is given as

$$y(x) = 24x - 9x^2. \quad (23)$$

Example 2 [31]:

$$\mathbf{y(x) = x + \int_0^1 (xt^2 + tx^2)y(x)dt}, \quad \mathbf{(24)}$$

Exact solution of (24) is given as

$$y(x) = \frac{80}{119}x^2 + \frac{180}{119}x. \quad (25)$$

Example 3 [23]:

$$y(x) = sin(2\pi x) + \int_0^1 cos(x)ydt, \quad (26)$$

Exact solution of(26) is given as

$$y(x) = sin(2\pi x). \quad (27)$$

The recorded results are displayed in Tables 1 to 3. By overserving the iteration size, computational time (in seconds) and maximum absolute error (MAE), this study can develop valid assumptions on the outcomes of three proposed methods over the problem (2). As we observe, Table 1 up to 3 shows that the iteration size and computational time of the SOR iterative method are shorter than the GS iterative method. However, the maximum absolute error of these iterative methods is increasing with respect to the increase in grid size N.

According to a recent study [27], the formulation of reduction percentage was introduced to compare the performance of three presented iterative methods. This formula explained how to compare the significant differences in the outcomes based on iteration size and computational time (in seconds). To make comparisons, we define the reduction percentage of the SOR method, ζ which can be formulated as

$$\zeta = {}^{(GS - SOR)}/_{GS} \times 100 \quad (28)$$

By applying the formula in Eq. (28) overall recorded results, we collect the reduction percentage SOR method in Table 4.

5.2. Definition [24, 25]

There are vector norm and matrix norm that we often refer to because it plays an important role in numerical analysis.

Definition 5.2.1

A norm on a vector space $\Re^{m\times n}$ is a mapping that associates with each vector x, a real number $\|x\|$ called the norm of x, such as the following properties are satisfied for all vectors x and y all scalars α:

I. $\|x\| \geq 0$and $\|x\| = 0$ if and only if $x = 0$ (positive definiteness)

II. $\|\alpha x\| = |\alpha|\|x\|$for all $x \in \Re n, \alpha \in \Re$ (absolute homogeneity)

III. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

A matrix norm $\Re^{m\times n}$is a mapping that associates with each $m \times n$ matrix A, a real number $\|A\|$, called the norm of A, such that the following properties are satisfied for all $m \times n$matrices A and B all scalars α:

Definition 5.2.2

I. **$\|A\| \geq 0$and $\|A\| = 0$ if and only if $A = 0$ (positive definiteness)**

II. **$\|\alpha A\| = |\alpha|\|A\|$for all $A \in \Re n, \alpha \in \Re$ (absolute homogeneity)**

III. **$\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)**

5.3. Results

According to Table 1 to Table 3, the results portrayed a good iteration size, computational time and maximum absolute error. The SOR family's iteration size and computational time are smaller than GS as the SOR method is modified by the GS formula with faster convergence speed in the iteration process compared to the GS method.

Table 1. The iteration size (IC) for all examples of FIE of the second kind

Example	N	Iteration Size (IC)	
		GS	SOR
1	256	213	43 (w=1.546)
	512	217	44 (w=1.553)
	1024	220	44 (w=1.551)
	2048	223	45 (w=1.552)
	4096	226	45 (w=1.551)
2	256	22	14 (w=1.121)
	512	23	14 (w=1.121)
	1024	23	14 (w=1.121)
	2048	23	14 (w=1.121)
	4096	24	14 (w=1.121)
3	256	71	27 (w=1.361)
	512	73	27 (w=1.361)
	1024	74	28 (w=1.361)
	2048	75	28 (w=1.361)
	4096	76	28 (w=1.361)

Table 2. The computational time (CT) for all examples of FIE of the second kind

Example	N	Computational time (CT)	
		GS	SOR
1	256	0.8004	0.5657
	512	2.5426	2.5425
	1024	10.6353	10.5422
	2048	46.7145	46.4116
	4096	263.9749	252.4253
2	256	0.74362	0.5990
	512	2.5984	2.5568
	1024	10.7259	10.5866
	2048	46.6486	46.5324
	4096	263.2550	262.1220
3	256	0.6379	0.6360
	512	2.7905	2.7732
	1024	11.2601	8.9257
	2048	50.2371	49.6225
	4096	277.1251	275.8081

Table 3. The Maximum absolute error (MAE) for all examples of FIE of the second kind

Example	N	Max. abs. error (MAE)	
		GS	SOR
1	256	3.96E-04	3.96E-04
	512	9.90E-05	9.90E-05
	1024	2.48E-05	2.48E-05
	2048	6.19E-06	6.19E-06
	4096	1.55E-06	1.55E-06
2	256	3.09E-06	3.09E-06
	512	7.72E-07	7.72E-07
	1024	1.93E-07	1.93E-07
	2048	4.82E-08	4.82E-08
	4096	1.20E-08	1.20E-08
3	256	2.11E-11	1.68E-12
	512	1.12E-11	1.70E-12
	1024	8.20E-12	9.16E-13
	2048	5.93E-12	9.20E-13
	4096	4.26E-12	9.21E-13

Table 4. The reduced percentage of SOR for all examples

Example	Iteration size (IC) %	Computational time (CT) %
1	79.3722 - 79.8122	0.0039 - 29.3228
2	36.3636 - 39.1304	0.2491 - 19.4373
3	61.9718 - 63.1579	0.2837 - 20.7316

Additionally, by seeing Table 4, the reduction percentage of iteration size and computational time of SOR shows that the SOR method has a good significant difference. It is seen that the reduction percentage of iteration size obtained by examples 1, 2 and 3 of SOR shows huge significant differences with 79.3722%- 79.8122%, 36.3636%-39.1304%, 61.9718%-63.1579%, respectively. Meanwhile, the computational time obtained by examples 1, 2 and 3 shows a reduction percentage with slight significant differences with 0.0039%-29.3228%, 0.2491%-19.4373%, and 0.2837%-20.7316%, respectively. Overall, the results of the proposed SOR iterative method are more efficient than GS iterative methods.

Fig. 6 Iteration size and computational time of GS and SOR for Problem 1

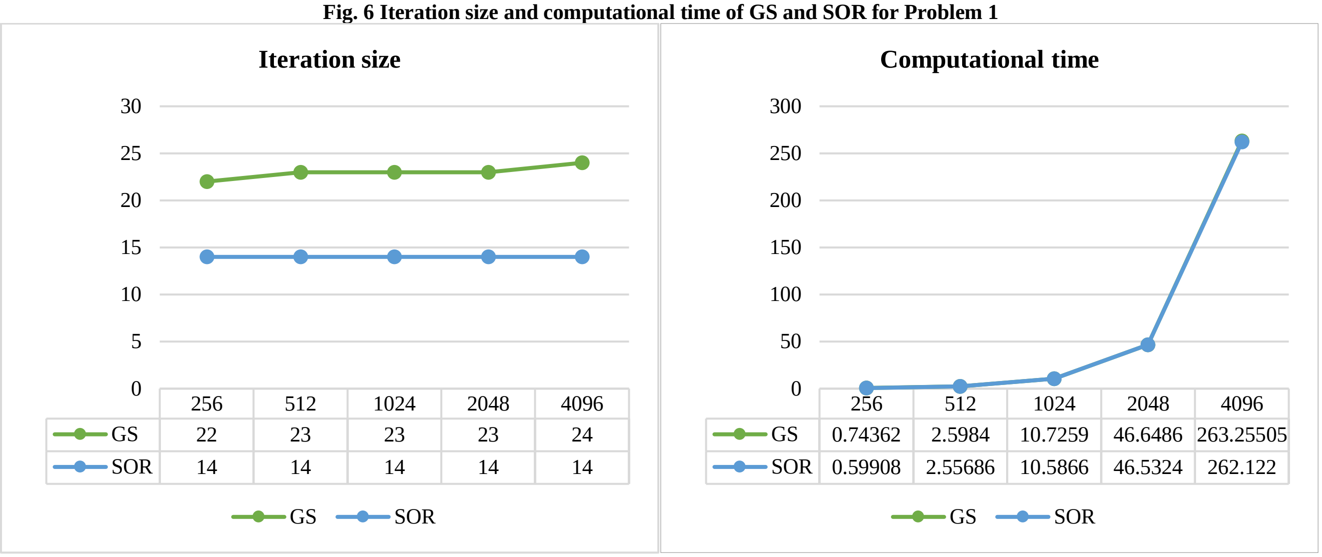

Fig. 7 Iteration size and computational time of GS and SOR for Problem 2

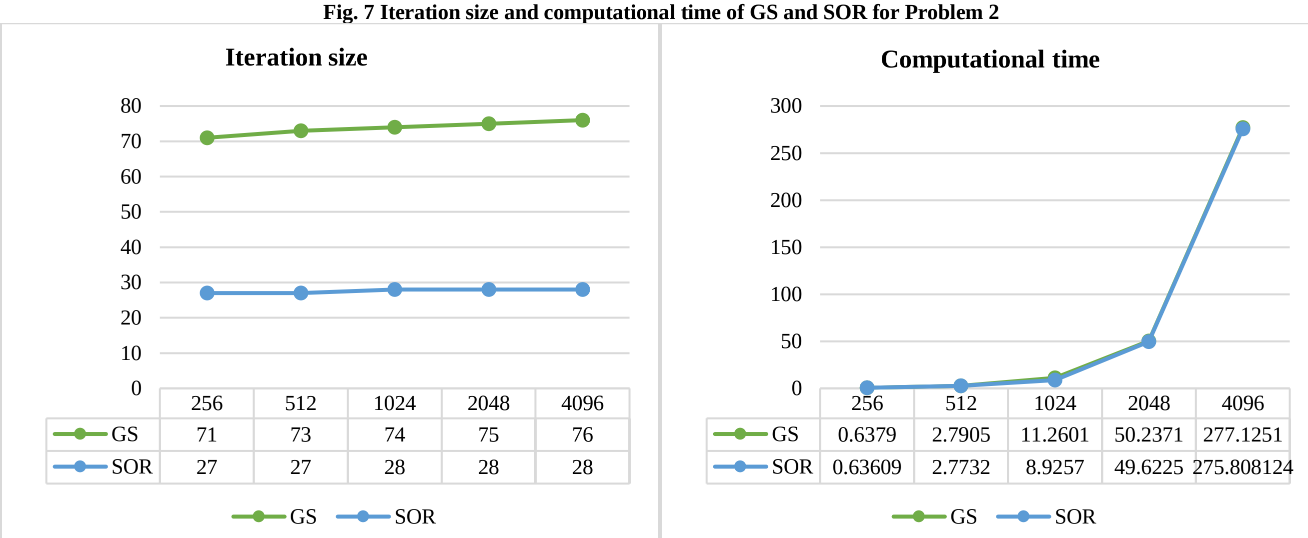

Fig. 8 Iteration size and computational time of GS and SOR for Problem 3

6. Conclusion

Therefore, this study concludes that based on the results, the first-order piecewise polynomial with collocation scheme in solving Fredholm integral equation of second kind with a full-sweep case of SOR is better in terms of iteration size, computational time due to the better speed of converge in iteration process with less of operational complexity. These results show that the previous study of the first-order piecewise polynomial and quadrature trapezoidal method improved the speed of convergence in terms of time and increased the accuracy of max absolute error. This is matched what has been explained in the literature review.

In summary, this study has proven that the SOR iteration is smaller than the GS iteration. Besides, the rate of convergence of the SOR iteration is faster than the GS iteration. This verifies that the SOR iteration has the best performance in solving the first-order polynomial piecewise on the Fredholm integral of the second kind.

Hence, future work for the SOR family has the potential to improve due to the benefits of the iterative method itself, that have weighted parameter. The study can expand the research of the SOR family by expanding the implementation of the improved quadrature method together with the SOR method on Fredholm integral equation of second kind [26, 27, 28, 29] beside the composited trapezoidal with SOR iterative method.

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